MIDTERM SOLUTION - PHY 402 - fall 12

PROBLEM I

1) Since $\hat{H}|a\rangle = \frac{E_0}{2}|a\rangle + \frac{\sqrt{3}E_0}{2}|b\rangle$

$\hat{H}|b\rangle = -\frac{E_0}{2}|b\rangle + \frac{\sqrt{3}E_0}{2}|a\rangle$

$$\hat{H}\left(\frac{\sqrt{3}|a\rangle + |b\rangle}{2}\right) = \frac{\sqrt{3}}{2}\left(\frac{E_0}{2}|a\rangle + \frac{\sqrt{3}}{2}E_0|b\rangle\right) + \frac{1}{2}\left(-\frac{E_0}{2}|b\rangle + \frac{\sqrt{3}}{2}E_0|a\rangle\right)$$

$$= E_0\left[\frac{\sqrt{3}}{2}|a\rangle + \frac{1}{2}|b\rangle\right]$$

$$\hat{H}\left(\frac{|a\rangle - \sqrt{3}|b\rangle}{2}\right) = \frac{1}{2}\left(\frac{E_0}{2}|a\rangle + \frac{\sqrt{3}}{2}E_0|b\rangle\right) - \frac{\sqrt{3}}{2}\left(-\frac{E_0}{2}|b\rangle + \frac{\sqrt{3}}{2}E_0|a\rangle\right)$$

$$= -E_0\left(\frac{|a\rangle - \sqrt{3}|b\rangle}{2}\right)$$

2) Since $\hat{H}\left(\frac{\sqrt{3}|a\rangle + |b\rangle}{2}\right) = E_0\left(\frac{\sqrt{3}|a\rangle + |b\rangle}{2}\right)$, $\frac{\sqrt{3}|a\rangle + |b\rangle}{2}$ is an eigenvector of $\hat{H}$ w/ eigenvalue E_0.

Similarly, $\frac{|a\rangle - \sqrt{3}|b\rangle}{2}$ is an eigenstate w/ eigenvalue $-E_0$

$$\boxed{E_0, -E_0}$$

3) Using the eigenstates of H $|1\rangle = \frac{\sqrt{3}|a\rangle + |b\rangle}{2}$, $|2\rangle = \frac{|a\rangle - \sqrt{3}|b\rangle}{2}$, The initial condition can be written as

$|\psi(0)\rangle = |a\rangle = \langle 1|a\rangle|1\rangle + \langle 2|a\rangle|2\rangle = \frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2}|2\rangle$. The energy eigenstates $|1,2\rangle$ evolve in time as $|1,2,t\rangle = e^{-iE_{1,2}t/\hbar}|1,2\rangle$ so, since the Schrödinger eq. is linear

$$|\psi(t)\rangle = \frac{\sqrt{3}}{2}e^{-iE_0t/\hbar}|1\rangle + \frac{1}{2}e^{+iE_0t/\hbar}|2\rangle = \longrightarrow$$

$$\boxed{\begin{aligned}|\psi(t)\rangle &= e^{-iE_0t/\hbar}\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}|a\rangle + \frac{1}{2}|b\rangle\right) + e^{iE_0t/\hbar}\frac{1}{2}\left(\frac{1}{2}|a\rangle - \frac{\sqrt{3}}{2}|b\rangle\right)\\ &= \left(\frac{3}{4}e^{-iE_0t/\hbar} + \frac{1}{4}e^{iE_0t/\hbar}\right)|a\rangle + \left(\frac{\sqrt{3}}{4}e^{-iE_0t/\hbar} - \frac{\sqrt{3}}{4}e^{iE_0t/\hbar}\right)|b\rangle\end{aligned}}$$

4) $$|\psi(t)\rangle = \underbrace{e^{-iE_0t/\hbar}\frac{\sqrt{3}}{2}}_{\text{prob. amplitude of } E_0}|1\rangle + \underbrace{\frac{1}{2}e^{iE_0t/\hbar}}_{\text{prob. ampl. of } -E_0}|2\rangle$$

eigenstates of $\hat{H}$

$$\boxed{\begin{aligned}&\text{probability of } E_0: \left|e^{-iE_0t/\hbar}\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}\\ &\text{probability of } -E_0 = \left|e^{iE_0t/\hbar}\frac{1}{2}\right|^2 = \frac{1}{4}\end{aligned}}$$

5) After the measurement the wave function collapses to the ground state $|2\rangle$ (assuming $E_0>0$, otherwise the ground state would be $|1\rangle$). That states evolves just by a phase $|\psi(t)\rangle = e^{+iE_0t/\hbar}|2\rangle$ and never acquires a component along $|1\rangle$. Any further measurement of the energy will give the ground state energy w/ 100% probability

$$\boxed{\text{prob. of ground state} = 100\%}$$

PROBLEM II

6) ~~The ground~~ The energy of the system will be the sum of the energy of each particle. Thus, in the ground state, both particles will be in the single-particle ground state. That means the orbital part of the wave function symmetric under the exchange of particle positions and, since the particles are fermions, the spin part must be anti-symmetric. The anti-symmetric combination of spin 1/2 has total spin $S=0$:

$$\frac{|1/2\, -1/2\rangle - |-1/2\, 1/2\rangle}{\sqrt{2}} = |s=0, m_s=0\rangle$$

$$\boxed{\text{spin } S=0}$$

7) $$\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\,\hat{S}_1 \cdot \hat{S}_2 \Rightarrow \hat{S}_1 \cdot \hat{S}_2 = \frac{\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2}{2}$$

$$\langle s=0, m=0 |\, \hat{S}_1 \cdot \hat{S}_2 \,| s=0, m=0\rangle = \langle s=0, m=0 |\, \frac{\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2}{2} \,| s=0, m=0\rangle$$

$$= \langle s=0, m=0 | \left[\frac{0(0+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}{2}\right] | s=0, m=0\rangle$$

$$\Rightarrow \boxed{\langle s=0, m=0 |\, \hat{S}_1 \cdot \hat{S}_2 \,| s=0, m=0\rangle = -\frac{3}{4}}$$

$$\langle s=1, m=1 |\, \hat{S}_1 \cdot \hat{S}_2 \,| s=1, m=1\rangle = \langle s=1, m=1 |\, \frac{\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2}{2} \,| s=1, m=1\rangle$$

$$= \langle s=1, m=1 | \left[\frac{1(1+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}{2}\right] | s=1, m=1\rangle$$

$$\Rightarrow \boxed{\langle s=1, m=1 |\, \hat{S}_1 \cdot \hat{S}_2 \,| s=1, m=1\rangle = \frac{1}{4}}$$

particle 1 particle 2

ground state : $|\text{ground}\rangle = |1\rangle \otimes |1\rangle \otimes |s=0, m=0\rangle$

single particle ground state

The shift in energy is given by 1st order pert. Theory as

$$\Delta E = \langle \text{ground}| -g\, \hat{S}_1 \cdot \hat{S}_2 \int dx\, |x,x\rangle\langle x,x|\, |\text{ground}\rangle$$

$$= -g \underbrace{\langle s=0, m=0|\, \hat{S}_1 \cdot \hat{S}_2\, |s=0, m=0\rangle}_{-\frac{3}{4}} \langle 1| \otimes \langle 1| \int dx\, |x\rangle \otimes |x\rangle\, \langle x| \otimes \langle x|\, |1\rangle \otimes |1\rangle$$

$$= \frac{3g}{4} \int dx\, \underbrace{\langle 1|x\rangle \langle x|1\rangle\ \langle 1|x\rangle \langle x|1\rangle}_{|\langle x|1\rangle|^4}$$

$$= \frac{3g}{4} \int_0^L dx \left[\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)\right]^4 = \frac{3g}{4} \frac{4}{L^2} \frac{3L}{8} \boxed{= \frac{9}{8} \frac{g}{L}}$$

g) For any state w/ total spin 1 (symetric in spin), the orbital part must be anti-symmetric under the exchange of particles. Thus they have the form $|\psi\rangle = \frac{|a\rangle \otimes |b\rangle - |b\rangle \otimes |a\rangle}{\sqrt{2}} \otimes |s=1, m=1\rangle$, for some $|a\rangle$ and $|b\rangle$. The 1st excited state will have $|a\rangle = |1\rangle$ and $|b\rangle = |2\rangle$. Computing the perturbation as above we have

$$\langle \text{excited}| -g\, \hat{S}_1 \cdot \hat{S}_2\, |\text{excited}\rangle = -g \langle s=1, m=1|\, \hat{S}_1 \cdot \hat{S}_2\, |s=1, m=1\rangle$$

$$\left[\frac{\langle 1| \otimes \langle 2| - \langle 2| \otimes \langle 1|}{\sqrt{2}}\right] \int dx\, |x\rangle \otimes |x\rangle\, \langle x| \otimes \langle x|$$

$$\left[\frac{|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle}{\sqrt{2}}\right]$$

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$$= \frac{3g}{4} \int dx \underbrace{\langle 1|x\rangle \langle x|1\rangle \;\; \langle 1|x\rangle \langle x|1\rangle}_{|\langle x|1\rangle|^4}$$

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$$= -g \underbrace{\langle s=0, m=0|\, \hat{S}_1 \cdot \hat{S}_2 \,|s=0, m=0\rangle}_{-\frac{3}{4}} \; \langle 1| \otimes \langle 1| \int dx \, |x\rangle \otimes |x\rangle \, \langle x| \otimes \langle x| \, |1\rangle \otimes |1\rangle$$

$$= \frac{3g}{4} \int dx \; \underbrace{\langle 1|x\rangle \langle x|1\rangle \; \langle 1|x\rangle \langle x|1\rangle}_{|\langle x|1\rangle|^4}$$

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$$\left[\frac{\langle 1| \otimes \langle 2| - \langle 2| \otimes \langle 1|}{\sqrt{2}} \right] \int dx \, |x\rangle \otimes |x\rangle \, \langle x| \otimes \langle x|$$

$$\left[\frac{|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle}{\sqrt{2}} \right]$$

$$= -\frac{g}{4}\frac{1}{4}\int dx \Big[\langle 1|x\rangle \langle 2|x\rangle \langle x|1\rangle \langle x|2\rangle + \langle 2|x\rangle \langle 1|x\rangle \langle x|2\rangle \langle x|1\rangle$$
$$- \langle 2|x\rangle \langle 1|x\rangle \langle x|1\rangle \langle x|2\rangle - \langle 1|x\rangle \langle 2|x\rangle \langle x|2\rangle \langle x|1\rangle \Big]$$

$\boxed{= 0}$

This reflects the fact that the perturbation acts only when the two particles are on top of each other. But in a state w/ anti-symmetric wavefunction this never happens.