Any question? contact: haizhao@umd.edu

QPI, #3

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

More Explanation

$\boxed{\text{"Clock" Rule}}$ x y z

i j k

A. Pauli Matrices, ε-tensor

(i) $[\sigma_i, \sigma_j]$ commutator

$= \sigma_i \sigma_j - \sigma_j \sigma_i$

Take one case as example, $[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2i\sigma_z$$

The same way, we can get $[\sigma_y, \sigma_z] = 2i\sigma_x$

$$[\sigma_z, \sigma_x] = 2i\sigma_y$$

$$[\sigma_y, \sigma_x] = -2i\sigma_z$$

$$[\sigma_z, \sigma_y] = -2i\sigma_x$$

$$[\sigma_x, \sigma_z] = -2i\sigma_y$$

$$\Rightarrow [\sigma_i, \sigma_j] = 2i\sum_k \varepsilon_{ijk}\sigma_k$$

(ii) $\{\sigma_i, \sigma_j\}$ anticommutator

$= \sigma_i\sigma_j + \sigma_j\sigma_i$

Take $\{\sigma_x, \sigma_y\}$ as example: $\{\sigma_x, \sigma_y\} = \sigma_x\sigma_y + \sigma_y\sigma_x$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 = 2\delta_{xy}$$

$$\{\sigma_x, \sigma_x\} = \sigma_x\sigma_x + \sigma_x\sigma_x$$

$$= 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\delta_{xx}$$

The same way. $\{\sigma_y, \sigma_z\} = \{\sigma_z, \sigma_x\} = \{\sigma_z, \sigma_y\} = \{\sigma_x, \sigma_z\} = \{\sigma_y, \sigma_x\} = 0$

$$\{\sigma_y, \sigma_y\} = \{\sigma_z, \sigma_z\} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \{\sigma_i, \sigma_j\} = 2\delta_{ij}$

(iii) $\sigma_i \sigma_j = \delta_{ij} + i \sum_k \epsilon_{ijk} \sigma_k$

$\sigma_x \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$\sigma_x \sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z$

$\sigma_y \sigma_x = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\sigma_z$

The same way., $\sigma_y \sigma_y = 1$

$\sigma_z \sigma_z = 1$

$\sigma_y \sigma_z = i\sigma_x$

$\sigma_z \sigma_x = i\sigma_y$

$\sigma_z \sigma_y = -i\sigma_x$

$\sigma_x \sigma_z = -i\sigma_y$

$\Rightarrow \sigma_i \sigma_j = \delta_{ij} + i \sum_k \epsilon_{ijk} \sigma_k$

More explanation

$i \to x$

$j \to y$

$\sigma_i \sigma_j = \sigma_x \sigma_y$

$= \delta_{ij} + i \sum_k \epsilon_{ijk} \sigma_k$

$= \delta_{xy} + i \sum_k \epsilon_{xyk} \sigma_k$

when $k = x$ or $k = y$, $\epsilon_{xyk} = 0$

so k can only be z

$= \delta_{xy} + i \cdot 1 \cdot \sigma_z$

$= i\sigma_z$

(iv) $tr(\sigma_i)$ trace of a matrix

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \text{Trace}(\sigma_x) = 0$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \text{Trace}(\sigma_y) = 0$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{Trace}(\sigma_z) = 0$

$\Rightarrow Tr(\sigma_i) = 0$

(v.) $tr(\sigma_i \sigma_j) = 2\delta_{ij}$

$$\sigma_x \sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow Tr(\sigma_x \sigma_y) = 0$$

$$\sigma_x \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow Tr(\sigma_x \sigma_x) = 2$$

The same way, $Tr(\sigma_y \sigma_y) = Tr(\sigma_z \sigma_z) = 2$

$$Tr(\sigma_x \sigma_y) = Tr(\sigma_y \sigma_z)$$
$$= Tr(\sigma_y \sigma_x) = Tr(\sigma_z \sigma_y)$$
$$= Tr(\sigma_x \sigma_z) = Tr(\sigma_z \sigma_x)$$
$$= 0$$

$$\Rightarrow tr(\sigma_i \sigma_j) = 2\delta_{ij}$$

More Explanation:

Easily, we can get this conclusion From (iii) & (iv)

When $i \neq j$:

$tr(\sigma_i) = 0$

$\sigma_i \sigma_j = (\pm) i \sigma_k \Rightarrow Tr(\sigma_i \sigma_j) = 0$

When $i = j$

$\sigma_i \sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow Tr(\sigma_i \sigma_i) = 2$

(vi) $\vec{v} \cdot \sigma \; \vec{w} \cdot \sigma = \vec{v} \cdot \vec{w} + i(\vec{v} \times \vec{w}) \cdot \sigma$

Proof: $\vec{\sigma} = \sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$ $\boxed{\text{Notice}}$

left:

① $(v_x \; v_y \; v_z) \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = v_x \sigma_x + v_y \sigma_y + v_z \sigma_z$

② $(w_x \; w_y \; w_z) \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = w_x \sigma_x + w_y \sigma_y + w_z \sigma_z$

$$①\cdot② = (v_x \sigma_x + v_y \sigma_y + v_z \sigma_z)(w_x \sigma_x + w_y \sigma_y + w_z \sigma_z)$$

$$= v_x w_x \sigma_x \sigma_x + \boxed{v_x w_y \sigma_x \sigma_y} + \boxed{v_x w_z \sigma_x \sigma_z}$$
$$+ \boxed{v_y w_x \sigma_y \sigma_x} + v_y w_y \sigma_y \sigma_y + \boxed{v_y w_z \sigma_y \sigma_z}$$
$$+ \boxed{v_z w_x \sigma_z \sigma_x} + \boxed{v_z w_y \sigma_z \sigma_y} + v_z w_z \sigma_z \sigma_z$$

Since from (iii) $\gamma_x \gamma_x = \gamma_y \gamma_y = \gamma_z \gamma_z = 1$

The three $\bigcirc$ terms,

$$= V_x W_x + V_y W_y + V_z W_z$$

$$= \vec{V} \cdot \vec{W}$$

The six $\square$ terms,

Following the "clock rule" in part (iii)

$$\begin{cases} \gamma_x \gamma_y = i \gamma_z \\ \gamma_y \gamma_x = -i \gamma_z \end{cases}$$

$$\begin{cases} \gamma_y \gamma_z = i \gamma_x \\ \gamma_z \gamma_y = -i \gamma_x \end{cases}$$

$$\begin{cases} \gamma_z \gamma_x = i \gamma_y \\ \gamma_x \gamma_z = -i \gamma_y \end{cases}$$

The $\square$ terms $= i V_x W_y \gamma_z - i V_x W_z \gamma_y$

$$- i V_y V_x \gamma_z \qquad + i V_y W_z \gamma_x$$

$$+ V_z W_x \gamma_y + i V_z W_y \gamma_x$$

$$= i \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{pmatrix} \begin{pmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix}$$

$$= i (\vec{V} \times \vec{W}) \gamma$$

$$\Rightarrow \vec{V} \cdot \gamma \, \vec{W} \gamma = \vec{V} \cdot \vec{W} + i (\vec{V} \times \vec{W}) \gamma$$

(vii) $\vec{V} \cdot \vec{W} = (V_x \ V_y \ V_z) \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} = V_x W_x + V_y W_y + V_z W_z$

$$= \sum_i V_i W_i$$

(viii) $\vec{v} \times \vec{w} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{pmatrix}$

$(\vec{v} \times \vec{w})_k = (V_x W_y - V_y W_x)\hat{k}$

when $i \to x$, $j \to y$ $\quad \varepsilon_{ijk} V_i W_j = V_x W_y$

when $i \to y$, $j \to x$ $\quad \varepsilon_{ijk} V_i W_j = -V_y W_x$

$\Rightarrow (\vec{v} \times \vec{w})_k = \sum_{ij} \varepsilon_{ijk} V_i W_j$

(ix) $\sum_k \varepsilon_{ijk} \varepsilon_{i'j'k'} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}$

We discuss the possible i, j, and i', j'

i	i'
j	j'

$\sum_k \varepsilon_{ijk} \varepsilon_{i'j'k}$

$i' = i$, $j' = j$ $\begin{cases} i = j & 0 \\ i \neq j & 1 \end{cases}$

$i' \neq i$, $i' = j$ $\begin{cases} j' = i & -1 \\ j' \neq i & 0 \end{cases}$

$i' \neq j$ $\quad 0$

$\Rightarrow \sum_k \varepsilon_{ijk} \varepsilon_{i'j'k'} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}$

x) $\vec{w} \times \vec{u} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_x & w_y & w_z \\ u_x & u_y & u_z \end{pmatrix}$

$$= \begin{pmatrix} w_y u_z - w_z u_y \\ w_z u_x - w_x u_z \\ w_x u_y - w_y u_x \end{pmatrix}$$

$$\vec{v} \times (\vec{w} \times \vec{u}) = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ \begin{pmatrix} w_y u_z \\ - w_z u_y \end{pmatrix} & \begin{pmatrix} w_z u_x \\ - w_x u_z \end{pmatrix} & \begin{pmatrix} w_x u_y \\ - w_y u_x \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} v_y w_x u_y - v_y w_y u_x - v_z w_z u_x + v_z w_x u_z \\ v_z w_y w_z - v_z w_z u_y - v_x w_x u_y + v_x w_y u_x \\ v_x w_z u_x - v_x w_x u_z - v_y w_y u_z + v_y w_z u_y \end{pmatrix}$$

$$= \begin{pmatrix} (v_y u_y + v_z u_z) w_x - (v_y w_y + v_z w_z) u_x \\ (v_x u_x + v_z u_z) w_y - (v_x w_x + v_z w_z) u_y \\ (v_x u_x + v_y u_y) w_z - (v_x w_x + v_y w_y) u_z \end{pmatrix}$$

$$= \begin{pmatrix} (v_x u_x + v_y u_y + v_z u_z) w_x - (v_x w_x + v_y w_y + v_z w_z) u_x \\ (v_x u_x + v_y u_y + v_z u_z) w_y - (v_x w_x + v_y w_y + v_z w_z) u_y \\ (v_x u_x + v_y u_y + v_z u_z) w_z - (v_x w_x + v_y w_y + v_z w_z) u_z \end{pmatrix}$$

$$= (\vec{v} \cdot \vec{u}) \vec{w} - (\vec{v} \cdot \vec{w}) \vec{u}$$

xi) $\vec{w} \times \vec{u} = \begin{pmatrix} W_y U_z - W_z U_y \\ W_z U_x - W_z U_z \\ W_y U_x - W_x U_y \end{pmatrix}$

$$\vec{v} \cdot (\vec{w} \times \vec{u}) = (V_x \; V_y \; V_z) \begin{pmatrix} W_y U_z - W_z U_y \\ W_z U_x - W_z U_z \\ W_y U_x - W_x U_y \end{pmatrix}$$

$$= V_x W_y U_z - V_x W_z U_y + V_y W_z U_x - \boxed{V_y W_z U_z} + V_z W_y U_x - V_z W_x U_y$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{pmatrix} U_x & U_y & U_z \end{pmatrix} \begin{pmatrix} V_y W_z - V_z W_y \\ V_z W_x - V_x W_z \\ V_x W_y - V_y W_x \end{pmatrix}$$

$$= U_x V_y W_z - U_x V_z W_y + U_y V_z W_x - U_y V_x W_z + U_z V_x W_y - \boxed{U_z V_y W_x}$$

Accordingly, we can find one by one

$$\Rightarrow \vec{V} \cdot (\vec{w} \times \vec{u}) = \vec{u} \cdot (\vec{v} \times \vec{w})$$

B.

(i) $|\psi\rangle\langle\psi|^n = |\psi\rangle\langle\psi|\psi\rangle\langle\psi|\psi\rangle\langle\psi| \cdots \langle\psi| = \hat{A}$

$$\cos(\lambda \hat{A}) = \sum_{n\text{-even}}^{\infty} \frac{\lambda^n (\hat{A})^n}{n!} = \hat{A} \sum_{n\text{-even}}^{\infty} \frac{\lambda^n}{n!}$$

When $\hat{A} = 1$, $\cos \lambda = \sum_{n\text{-even}}^{\infty} \frac{\lambda^n}{n!}$

$$\Rightarrow \cos(\lambda \hat{A}) = \hat{A} \cos \lambda$$

(ii) Any Hermitian operator $\hat{A}$ can be written as: $\hat{A} = \sum_n a_n |n\rangle\langle n|$

Proof:

Since $|n\rangle$ is eigenvector, and a_n is eigenvalue

$$\Rightarrow \quad \hat{A}|n\rangle = a_n|n\rangle$$

$$\langle\psi|\psi\rangle = \sum_n |n\rangle\langle n| = 1$$

$$\Rightarrow \hat{A} = \hat{A}\sum_n |n\rangle\langle n| = \sum_n a_n|n\rangle\langle n|$$

C. spin: $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$|+\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$|-\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$$

Rotate around z-axis, the operator $R_z(\phi) = \exp\left(-\frac{i S_z \phi}{\hbar}\right)$

$$= \begin{pmatrix}\exp(-\frac{i\phi}{2}) & 0\\ 0 & \exp(\frac{i\phi}{2})\end{pmatrix}$$

Here $\phi = \pi$. $R_z(\pi) = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$

$$|\psi_{new}\rangle = R_z|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\ -1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(-i|+\rangle - |-\rangle)$$

So the probability for being up $\frac{-i\,i}{(\sqrt{2})^2} = \frac{1}{2} = 50\%$

and down $\frac{(-1)^2}{(\sqrt{2})^2} = \frac{1}{2} = 50\%$