PHY 402 - HW2 - SOLUTION

A. i)

$$\langle p | y \rangle = \int dx \, \underbrace{\langle p | x \rangle}_{\underbrace{\langle x | p \rangle^*}_{\frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}}} \underbrace{\langle x | y \rangle}_{\delta(x-y)} = \frac{e^{-ipy/\hbar}}{\sqrt{2\pi\hbar}}$$

p: eigenvect of $\hat{p}$ w/ eigenvalue p

y: eigenvect of $\hat{x}$ w/ eigenvalue y

$$|y\rangle = \int dp \, |p\rangle \langle p | y \rangle = \int dp \, \underbrace{\frac{e^{-ipy/\hbar}}{\sqrt{2\pi\hbar}}}_{\text{coordinates of } y \text{ in the } |p\rangle \text{ basis}} |p\rangle$$

ii) matrix element:

$$\langle p | \hat{x} | p' \rangle = \int dx\, dx' \, \langle p | x \rangle \underbrace{\langle x | \hat{x} | x' \rangle}_{\underbrace{x' \langle x | x' \rangle}_{x' \, \delta(x - x')}} \langle x' |$$

$$= \int dx \, \underbrace{\langle p | x \rangle}_{\frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}} \, x \, \underbrace{\langle x | p' \rangle}_{\frac{e^{ip'x/\hbar}}{\sqrt{2\pi\hbar}}}$$

$$= \int dx \, x \, \frac{e^{-i(p-p')x/\hbar}}{\sqrt{2\pi\hbar}^{\,2}}$$

$$= i\hbar \frac{d}{dp} \underbrace{\int \frac{dx}{2\pi\hbar} e^{-i(p-p')x/\hbar}}_{\delta(p-p')}$$

$$= i\hbar \frac{d}{dp} \delta(p-p')$$

iii) $\langle x | \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 | \psi\rangle = \int dy \, \langle x | \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 | y\rangle \underbrace{\langle y | \psi\rangle}_{\psi(y)}$

but
$$\langle x | \hat{x} | y\rangle = \langle x | y | y\rangle = y \langle x|y\rangle = y\,\delta(x-y)$$
$$\langle x | \hat{x}^2 | y\rangle = \langle x | y^2 | y\rangle = y^2 \langle x|y\rangle = y^2\delta(x-y)$$
$$\langle x | \hat{p} | y\rangle = -i\frac{d}{dx}\delta(x-y) \quad \text{(as shown in class)}$$
$$\langle x | \hat{p}^2 | y\rangle \underset{\substack{\text{definition}\\ \text{of } \hat{p} \text{ in our}\\ \text{class}}}{=} -i\frac{d}{dx}\langle x|\hat{p}|y\rangle = -\frac{d^2}{dx^2}\langle x|y\rangle = -\frac{d^2}{dx^2}\delta(x-y)$$

so $\langle x | \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 | \psi\rangle = \int dy \left[-\frac{1}{2m}\frac{d^2}{dx^2}\delta(x-y) + \frac{m\omega^2}{2} y^2 \delta(x-y)\right]\psi(y)$

$$= -\frac{1}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{m\omega^2}{2}x^2\psi(x)$$

B. i) $\hat{H}|\psi\rangle = \Big[E_0\big(|A\rangle\langle A| + |B\rangle\langle B|\big) + T\big(|A\rangle\langle B| + |B\rangle\langle A|\big)\Big]\left(\frac{|A\rangle + |B\rangle}{\sqrt{2}}\right)$

$$= \big(E_0|A\rangle + E_0|B\rangle + T|A\rangle + T|B\rangle\big)/\sqrt{2}$$

$$= (E_0+T)\left(\frac{|A\rangle + |B\rangle}{\sqrt{2}}\right) = (E_0+T)|\psi\rangle$$

ii) The Time-evolution is more easily computed in the eigenbasis of $\hat{H}$. Since we already used $\hat{H}$ in class we know the eigenvalues/eigenkets

$$\hat{H}|+\rangle = (E_0+T)|+\rangle, \quad |+\rangle = \frac{|A\rangle + |B\rangle}{\sqrt{2}}$$
$$\hat{H}|-\rangle = (E_0-T)|-\rangle, \quad |-\rangle = \frac{|A\rangle - |B\rangle}{\sqrt{2}}$$

The Time evolution of $|+\rangle, |-\rangle$ in Time is Trivial:

$$i\hbar \frac{d}{dt}|+,t\rangle = \hat{H}|+,t\rangle \Rightarrow |+,t\rangle = e^{-i(E_0+T)t/\hbar}|+\rangle$$

$$i\hbar \frac{d}{dt}|-,t\rangle = \hat{H}|-,t\rangle \Rightarrow |-,t\rangle = e^{-i(E_0-T)t/\hbar}|-\rangle$$

Write $|\psi\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis:

$$|\psi\rangle = |+\rangle\langle+|\psi\rangle + |-\rangle\langle-|\psi\rangle \qquad a(0)=1 \quad b(0)=0$$
$$= \frac{|A\rangle+|B\rangle}{\sqrt{2}}\left(\frac{\langle A| + \langle B|}{\sqrt{2}}\right)(a|A\rangle + b|B\rangle)$$
$$+ \frac{|A\rangle-|B\rangle}{\sqrt{2}}\left(\frac{\langle A| - \langle B|}{\sqrt{2}}\right)(a|A\rangle + b|B\rangle)$$
$$= \frac{1}{2}(|A\rangle + |B\rangle)(a+b) + \frac{1}{2}(|A\rangle - |B\rangle)(a-b)$$
$$= \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$
$$\Downarrow$$
$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i(E_0+T)t/\hbar}|+\rangle + \frac{1}{\sqrt{2}}e^{-i(E_0-T)t/\hbar}|-\rangle$$
$$= e^{-iE_0t/\hbar}\left[\cos\frac{Tt}{\hbar}|A\rangle - i\sin\frac{Tt}{\hbar}|B\rangle\right]$$

iii) $E = E_0 + T$ w/ probability $|\langle + | \psi(t)\rangle|^2 = \frac{1}{2}$

$E = E_0 - T$ w/ probability $|\langle - | \psi(t)\rangle|^2 = \frac{1}{2}$

C. i) $|\psi\rangle = \int dp \, \langle p|\psi\rangle \, |p\rangle \Rightarrow$ probability (density) of measuring $p = p_0$ $= |\langle p_0|\psi\rangle|^2$

but $\langle p_0|\psi\rangle = \int dx \, \underbrace{\langle p_0|x\rangle}_{\langle x|p_0\rangle^*} \underbrace{\langle x|\psi\rangle}_{\psi(x)} = \int dx \, \frac{e^{-ip_0x/\hbar}}{\sqrt{2\pi\hbar}} \psi(x)$

For the ground state of the harm. oscillator $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$ so

$$\langle p_0|\psi\rangle = \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-ip_0x/\hbar - \frac{m\omega x^2}{2\hbar}}$$

$$= \frac{1}{\pi^{1/4}} \frac{1}{(m\hbar\omega)^{1/4}} e^{-p^2/2m\hbar\omega}$$

so probability density $= \frac{1}{(m\hbar\pi\omega)^{1/2}} e^{-\frac{p^2}{m\hbar\omega}}$

ii) The eigenfunction of momentum w/ eigenvalue p_0, that is, $|p_0\rangle$. In the x-basis it is:

$$\langle x|p_0\rangle = \frac{e^{ip_0x/\hbar}}{\sqrt{2\pi\hbar}}$$

NOTE: This is not a normalizable wavefunction so, strictly speaking, the particle can never be on this state and the momentum can never be infinitely well measured. In practice p is known w/ some experimental uncertainty and the wave function after the measurement is very broad but will tend to zero eventually as $|x|\to\infty$.

D. i) $$e^{-\frac{iy\hat{p}}{\hbar}} f(x) = e^{-y\frac{d}{dx}} f(x)$$

$$= \sum_{n=0}^{\infty} \frac{(-y)^n}{n!}\frac{d^n}{dx^n} f(x)$$

$$= f(x) - y f'(x) + \frac{y^2}{2} f''(x) - \frac{y^3}{6} f'''(x) + \dots$$

$$= f(x-y)$$

ii) $$\underbrace{\underset{\langle x|\psi(t)\rangle}{\psi(x,t)} = e^{-i\hat{H}t/\hbar} \underset{\langle x|\psi(0)\rangle}{\psi(x,0)}}_{\text{in the } |x\rangle \text{ basis}} \Rightarrow \underbrace{|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle}_{\text{True in any basis}}$$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar\frac{d}{dt}e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \hat{H}e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \hat{H}|\psi(t)\rangle$$

NOTE: how do I know that $\frac{d}{dt}e^{-i\hat{H}t/\hbar} = -\frac{i\hat{H}}{\hbar}e^{-i\hat{H}t/\hbar}$? $\longrightarrow$

Using the definition of exponentials:

$$e^{-i\hat{H}t/\hbar} = \sum_{n=0}^{\infty} \left(\frac{-i\hat{H}t}{\hbar}\right)^n \frac{1}{n!} = 1 - \frac{it}{\hbar}\hat{H} - \frac{t^2}{2\hbar^2}\hat{H}^2 + \dots$$

$$\frac{d}{dt} e^{-i\hat{H}t/\hbar} = -\frac{i}{\hbar}\hat{H} - \frac{t}{\hbar^2}\hat{H}^2 + \dots = -\frac{i\hat{H}}{\hbar}\left[1 - \frac{it}{\hbar}\hat{H} + \dots\right]$$

$$= -\frac{i}{\hbar}\hat{H}\, e^{-i\hat{H}t/\hbar}$$