KA's & KET's

(or linear algebra in Dirac's notation)

- KET's \((\psi')\) are vectors, except they live in spaces with dimensions \(\neq 3\) - we could write \(\psi\), but we don't, we write \(\langle\psi'\rangle\)

- The other difference is that KET's live in a complex linear space, that is, they can be multiplied by complex numbers:

  \[
  \alpha \langle\psi'\rangle = \langle\psi''\rangle \quad \alpha = \text{KET}
  \]

- Just like regular vectors, sometimes we want to do calculations in a specific coordinate system (or basis). We can write a KET in terms of a basis \(\epsilon \{1 \rangle, \langle 2 \rangle, \ldots \rangle\) as

  \[
  \langle\psi'\rangle = \sum \langle \psi' | \epsilon_n \rangle \langle \epsilon_n | \rangle
  \]

  \[
  \langle \psi' | \epsilon_n \rangle = \text{coordinates of } \langle\psi'\rangle \text{ in the } \epsilon \{1 \rangle, \langle 2 \rangle, \ldots \rangle \text{ basis}
  \]

  \[
  \langle \psi' | \epsilon_n \rangle = \langle \psi' | \epsilon_n \rangle^* \quad \text{don't forget!}
  \]

- The scalar product can be a complex number. Also, in a complex space

  \[
  \langle \psi' | \psi'' \rangle = \langle \psi' | \psi'' \rangle^* \quad \text{don't forget!}
  \]
- Frequently, we want to do explicit calculations using a coordinate system (or basis). Orthonormal basis have the property:

\[ \langle \hat{n} | m \rangle = \delta_{nm} \]

complete:

\[ \sum_n |n\rangle \langle n| = I \]

projects a ket as the direction \( |m\rangle \):

\[ |m\rangle \langle n| = \delta_{mn} |m\rangle \]

The same ket can be represented in different basis:

\[ |\psi\rangle = \sum_n |n\rangle \langle n| |\psi\rangle = \sum_n \langle n| \psi \rangle |n\rangle \]

\[ \psi_n = \langle n| \psi \rangle \]

Linear operators (take a ket and produce another ket) also have coordinates:

\[ A |\psi\rangle = |\psi\rangle \Rightarrow \langle \psi|A|\psi\rangle = \sum_n \langle n| \psi \rangle \langle A |n\rangle = \sum_n \langle n| \psi \rangle A_{nm} \langle m| \psi \rangle = \sum_n \langle n| \psi \rangle \langle n| A |m\rangle \langle m| \psi \rangle = \sum_n A_{nm} \langle n| \psi \rangle \langle m| \psi \rangle = \sum_n A_{nm} \delta_{nm} = \sum_n A_{nm} \delta_{nm} = \sum_n \delta_{nm} |\psi\rangle \]

\[ = |\psi\rangle \]

\[ \psi_n = \langle n| \psi \rangle \]

\[ A_{nm} = \langle n| A |m\rangle \]

\[ A = \sum_{nm} A_{nm} |n\rangle \langle m| \]

\[ = A_{nm} |n\rangle \langle m| \]

\[ A_{nm} \]
• We can give a meaning to \( \langle \Psi | \Psi \rangle \) by itself. It's something that takes a ket and produces a number, namely, the scalar product between |\( \Psi \rangle \) and the inner ket:

\[
\langle \Psi | (\langle \Psi | \Psi \rangle) = \langle \Psi | \Psi \rangle
\]

feed a ket get a number

bra ket bracket = number

• The rule relating a ket \( |\Psi \rangle \) to \( \langle \Psi | \) is denoted by a dagger and called Hermitian conjugation

\[
\langle \Psi | = |\Psi \rangle^\dagger
\]

Note that:

\[
(\alpha |\Psi \rangle)^\dagger |\Psi \rangle = [\langle \Psi | (\alpha |\Psi \rangle)]^\dagger
= [\alpha \langle \Psi | \Psi \rangle]^\dagger
= \alpha^* \langle \Psi | \Psi \rangle
= \alpha^* \langle \Psi | \Psi \rangle
\]

\[
(\alpha |\Psi \rangle)^\dagger = \alpha^* \langle \Psi |
\]
Sometimes we organize the components $\psi_n$ as a column and $A_{nm}$ as a matrix. Then $|\psi\rangle = \hat{A} |\psi\rangle$ turns into

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} & \cdots \\
A_{21} & A_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots
\end{pmatrix}
\]

Some of you only knew linear algebra in this matrix language. Now you know better.

By the way, the equation above explain why matrix multiplication is defined in that weird "row-times-column" manner.

- Hermitian operators are those satisfying
  \[ \langle \psi_1 | \hat{A} | \psi_2 \rangle = \langle \psi_2 | \hat{A} | \psi_1 \rangle^* \]
  Their matrix elements in a orthonormal basis satisfy
  \[ A_{nm} = \langle n | \hat{A} | m \rangle = \langle m | \hat{A} | n \rangle^* = A_{mn}^* \]
  or
  \[ A_{12} = A_{21}^* \quad A_{13} = A_{31}^* \quad A_{11} = \text{real} \quad A_{22} = \text{real} \quad \ldots \]

- The eigenvectors of a Hermitian operator form an orthonormal basis and its eigenvalues are real.
  \[ \hat{A} | n \rangle = a_n | n \rangle \]
  - $a_n$ is real
  - $\langle m | n \rangle = \delta_{mn}$ (orthonormal)
  - $\sum | n \rangle \langle n | = I$ (complete set)
For a particle moving on a (continuous) line, the eigenvalue of the position operator \( \hat{x} \) can be any real number.

\[ \hat{x} |x\rangle = x |x\rangle \]

We can use \( \{ |x\rangle \} \) as a basis.

\[ |y\rangle = \int dx |x\rangle \langle x|y\rangle = \int dx \psi(x) |x\rangle \]

We call this the wavefunction.

What is the wave function corresponding to an eigenstate of position?

The matrix elements of \( \hat{x} \) in the \( \{ |x\rangle \} \) basis are

\[ \langle x|\hat{x}|y\rangle = \langle x|y\rangle = y \]

\[ \delta(x-y) = x \delta(x-y) \]

Not \( \delta \) because \( x, y \) are continuous.

Let us write the eigenvalue (eigen-vector equation for \( \hat{x} \) in the \( \{ |x\rangle \} \) basis:

\[ \sum_{n=0}^{\infty} \psi_n = x \psi_n \]

\[ \int dx \delta(x-y) \psi(x) = x \psi_y(x) \]

\[ \Rightarrow \psi_x(y) = x \psi_y(x) \]

\[ \Rightarrow \psi_x(y) = \delta(x-y) \]
Scalar product in the $\{x\}$ basis:

$$\langle \psi | \psi \rangle = \langle \psi \downarrow | \psi \downarrow \rangle = \int dx \langle \psi(x) | \psi(x) \rangle$$

$$= \int dx \, \mathcal{K}(x) \psi(x)$$

I notice the similarity to $\psi(y) \mathcal{K}(x) \psi(x) + \cdots$.

* Another useful basis is formed by eigenfunctions of the momentum operator $\hat{\pi}$. In the $\{x\}$ basis $\hat{\pi}$ is defined by

$$\langle x \mid \hat{\pi} \mid y \rangle = -i \hbar \frac{d}{dx} \langle x \mid \psi \rangle = -i \hbar \frac{d}{dx} \langle x \mid \psi \rangle$$

$$\Rightarrow \langle x \mid \hat{\pi} \mid y \rangle = -i \hbar \delta(x - y)$$

What are the eigenfunctions of $\hat{\pi}$? In the $\{x\}$ basis they are given by:

$$\hat{\pi} \mid p \rangle = p \mid p \rangle \Rightarrow \langle x \mid \hat{\pi} \mid p \rangle = p \langle x \mid \psi \rangle$$

$$\Rightarrow -i \hbar \frac{d}{dx} \langle x \mid \psi \rangle = p \langle x \mid \psi \rangle$$

$$\Rightarrow \langle x \mid \psi \rangle = \frac{e^{ipx}}{\sqrt{2\pi\hbar}}$$

Normalization chosen so

$$\langle p \mid p \rangle = \delta(p - p')$$

Coordinates of $|\psi\rangle$ in the $\{x\}$ basis:

$$\langle x \rangle = \langle x \mid \hat{\pi} \rangle = \int dp \, \langle x \mid \psi \rangle \langle p \mid \psi \rangle = \int dp \, \frac{-i \hbar}{\sqrt{2\pi \hbar}} e^{ipx} \langle p \mid \psi \rangle$$

$$\langle p \mid \psi \rangle = \tilde{\psi}(p)$$

= Fourier Transform of $\langle \psi \rangle$
Let us now apply the basis set theory we learned to the measurement problem (postulate 3 of our review of QM). Given the ket \( |\psi\rangle \) describing the system at a time of measuring an observable \( A \), the rule giving the probability of finding the value \( a_n \) can be summarized as:

\[
|a_n| = \frac{\langle n|\psi\rangle}{\langle n|n\rangle}
\]

\[p_n = |\langle n|\psi\rangle|^2\]

Let us see how this work when measuring the position of a particle moving in 1 dimension. The relevant operator is \( \hat{x} \), with eigenvectors \( |x\rangle \). We can decompose the ket \( |\psi\rangle \) in this basis:

\[
|\psi\rangle = \int dx \langle x|\psi\rangle |x\rangle
\]

Postulate 3 says that the probability (actually, probability density) of finding the particle at some location \( x_0 \) is

\[
p(x) = |\langle x|\psi\rangle|^2 = |\langle x_0|\psi\rangle|^2 = |\langle x_0|\psi\rangle|^2
\]

Of course, this is the rule you learned in QM 1.
What if we measured the momentum $\hat{p}$? Then we'd have to expand $|\psi\rangle$ in the $|p\rangle$ basis.

$$|\psi\rangle = \int dp \frac{\langle p | \psi \rangle |p\rangle}{\tilde{\psi}(p)}$$

The probability (density) of finding the value $p_0$ when measuring the momentum is given by

$$P(p_0) = |\langle p_0 | \psi \rangle|^2 = |\tilde{\psi}(p_0)|^2,$$

again, as you learned in QM 1.