Lecture 32 Highlights

We are going to consider the problem of predicting the energy density of photons in a black body radiation source. The answer is the well-known Planck blackbody distribution function:

$$\rho(\omega) = \frac{\hbar\omega^3 / \pi^2 c^3}{e^{\hbar\omega/k_B T} - 1}$$

Note that this is an energy per unit frequency and per unit volume.

Where does this distribution come from? Can we derive it from our quantum statistical mechanics results?

Photons have 3 important properties:

1) They are spin-1 relativistic (massless) particles. The photon carries either one $\hbar$ of right-circularly polarized angular momentum, or one $\hbar$ of left-circularly polarized angular momentum. As such it is a Boson.
2) Photons carry energy $E = \hbar\omega = h\nu$, where ω is the angular frequency of the oscillating E and B fields, and ν is the linear frequency. This is known from Einstein's explanation of the photo-electric effect.
3) Photons carry momentum $p = \hbar k = \frac{h}{\lambda} = \frac{hf}{c} = \frac{E}{c}$. This is known from Compton scattering experiments. Note that the dispersion relation for light in free space is given by $\omega = kc$. This is the relation between the wavenumber and the frequency of the wave.

A gas of photons of equal energy constitutes a gas of identical Bosons. However in equilibrium the number of photons in the box is constantly fluctuating as they are absorbed and re-emitted by atoms in the walls of the box. In this case the particle number N is not fixed, and we release the number constraint by setting $\alpha = \mu = 0$.

The identical Bosons will obey Bose-Einstein statistics and have an occupation number distribution in equilibrium at temperature T given by;

$$n_s = \frac{g_s}{e^{+(E_s)/k_B T} - 1} \quad \text{Identical Photons}$$

Now the question is this: What are the energies of all the states, E_s and the degeneracies of all the states, g_s? This is the hard part…

We seek solutions to Maxwell's equations for the light in an empty cube of side $a \times a \times a = V$. We take the walls of the cube to lie on the Cartesian planes and put one corner of the box at the origin of the coordinate system. We shall assume that the 6 walls of the cube present perfect metal boundary conditions to the light. This means that the tangential component of $\vec{E}$ must be zero on all of the walls. Maxwell's equations can be combined to create a vector wave equation for the electric field in free space (i.e. no sources: $\rho = \vec{J} = 0$):

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

This is three equations in one, and we will focus for the moment on one component (which yields the Helmholtz equation): $\nabla^2 E_x + k^2 E_x = 0$, where $E_x = E_x(x, y, z)$. To solve it we use the same trick that was applied to the Schrödinger equation for the

Hydrogen atom, namely separation of variables: $E_x(x,y,z) = X(x)Y(y)Z(z)$.
Substituting this into the Helmholtz equation and dividing through by the product XYZ yields:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2$$

Since each term on the LHS is a function of only x, only y and only z, and the RHS is a constant, they must each separately equal a constant. We call these constants $-k_x^2$, etc. and end up with three essentially equivalent ordinary differential equations and an algebraic constraint:

$$X'' = -k_x^2 X \text{ , etc. and } k_x^2 + k_y^2 + k_z^2 = k^2$$

The solutions to the differential equations are sines and cosines:

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$
$$Y(y) = C\sin(k_y y) + D\cos(k_y y)$$
$$Z(z) = E\sin(k_z z) + F\cos(k_z z)$$

The boundary conditions force half of the coefficients to be zero, and force the wavenumbers to take on only special values:

$k_x = \dfrac{\ell\pi}{a}$, $k_y = \dfrac{m\pi}{a}$, and $k_z = \dfrac{n\pi}{a}$, where $\ell = 1,2,3,...$ and $m = 1,2,3,...$ and $n = 1,2,3,...$

Now the total momentum of the photon is quantized, according to:

$k^2 = \left(\dfrac{\ell\pi}{a}\right)^2 + \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{a}\right)^2$, where ℓ, m, n are positive integers.

We now know the energies of all the states of the electromagnetic fields in the box. They are given by

$E_{\ell mn} = \hbar c \dfrac{\pi}{a}\sqrt{\ell^2 + m^2 + n^2}$, where the index "s" has been replaced by a list of 'quantum numbers' ℓ, m, n.

Now, what are the degeneracies of these states? In other words, how many different lists of quantum numbers have the same energy? To do this calculation we move into "k-space" or "momentum space" or "reciprocal space." Each state of the electromagnetic fields in the box is labeled by a discrete point in momentum space given by the vector $\vec{k} = \dfrac{\ell\pi}{a}\hat{k}_x + \dfrac{m\pi}{a}\hat{k}_y + \dfrac{n\pi}{a}\hat{k}_z$. Many of the photons in the box are labeled by states very far from the origin of momentum space. The discrete states become almost a continuum at high values of (ℓ, m, n), and we can treat them as if they are continuous, to good approximation. We can find the degeneracy of the states by the following argument. All the states of the same energy will lie on (or very near) a spherical octant centered on the origin of momentum space. An estimate of the number of states in a spherical octant shell of thickness dk is: #states = Volume of shell / (volume per state) = $\dfrac{(1/8)\,4\pi k^2\, dk}{\pi^3/V} \times 2$, where the factor of 2 comes from two possible projections of the relativistic photon spin onto its direction of motion. We can refer to this result as

$g(k)\,dk$, the generalization of the degeneracy g_s to the case where the photon states are labeled by their wavenumber k.

We can re-express the degeneracy in terms of frequency using the fact that $\omega = kc$ as $g(\omega) = \frac{V}{\pi^2}\frac{\omega^2}{c^3}$. Now the occupation number of the photon states is given by $n(\omega) = \frac{g(\omega)}{e^{\hbar\omega/k_BT} - 1} = \frac{V\omega^2/(\pi^2 c^3)}{e^{\hbar\omega/k_BT} - 1}$. The final step is to calculate the energy density of the electromagnetic fields in the box ($\rho(\omega)$). This is found from the occupation number (n_s) times the energy of that occupied state ($\hbar\omega$), divided by the volume: $\rho(\omega) = \frac{\hbar\omega^3/(\pi^2 c^3)}{e^{\hbar\omega/k_BT} - 1}$, which is the Planck blackbody radiation formula. Hence the photons in the box act like a gas of identical Bosons of fixed total energy, but not fixed total number!