Lecture 14 Highlights

We considered the hyperfine interaction between the magnetic moment of the proton and that of the electron in the hydrogen atom.

The proton has a magnetic moment due to the intrinsic spin, orbital motion of its quark constituents, and the quark-gluon plasma. It is given by:

$$\vec{\mu}_p = \frac{ge}{2m_p}\vec{S}_p,$$

where $g \cong 5.59$, e is the electronic charge, m_p is the mass of the proton, and $\vec{S}_p$ is its spin angular momentum. Note that the proton is a spin-1/2 particle, just like an electron. Its spin angular momentum lives on a 2-state ladder, with steps separated by $\hbar$, just like the electron. In the hydrogen atom the magnetic field generated by the proton's magnetic moment interacts with the magnetic moment of the electron to give rise to the hyperfine perturbing Hamiltonian:

$$\mathrm{H}_{HF} = -\vec{\mu}_e \bullet \vec{B}_{dip},$$

where the magnetic field due to the proton's dipole moment is given by Griffiths E+M book, Eq. 5.90:

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{\mu}_p \bullet \hat{r})\hat{r} - \vec{\mu}_p\right] + \frac{2\mu_0}{3}\vec{\mu}_p \delta^3(\vec{r}).$$

The last term comes from the infinitesimal dipole at the proton location. This expression assumes that the proton is at the origin and is oriented in space in the direction of $\vec{\mu}_p$, and calculates the vector magnetic field at location $\vec{r}$.

Evaluating the first order correction to the energy of the hydrogen atom yields:

$$E_n^1 = \left\langle \psi_n^0 \middle| \mathrm{H}_{HF} \middle| \psi_n^0 \right\rangle$$

$$= \frac{\mu_0 g e^2}{8\pi m_e m_p}\left\langle \psi_n^0 \middle| \frac{3(\vec{S}_p \bullet \hat{r})(\vec{S}_e \bullet \hat{r}) - \vec{S}_e \bullet \vec{S}_p}{r^3} \middle| \psi_n^0 \right\rangle + \frac{\mu_0 g e^2}{3m_e m_p}\left\langle \psi_n^0 \middle| \vec{S}_e \bullet \vec{S}_p \delta^3(\vec{r}) \middle| \psi_n^0 \right\rangle$$

where ψ_n^0 is the un-perturbed hydrogen atom wavefunction and "n" in the wavefunction subscript represents a list of quantum numbers.

If we specialize to the case of zero orbital angular momentum, $\ell = 0$, for the un-perturbed states, the first term above is zero (see problem 6.27). The second term simplifies because of the delta function, and we have:

$$E_{n,0,0}^1 = \frac{\mu_0 g e^2}{3m_e m_p}\left\langle \vec{S}_e \bullet \vec{S}_p \right\rangle \left|\psi_{n,0,0}^0(0)\right|^2,$$

where now n represents the principal quantum number in the hydrogen atom. One finds from Eq. (4.89) that $\left|\psi_{n,0,0}^0(0)\right|^2 = \frac{1}{n^3}\frac{1}{\pi a^3}$, where a is the Bohr radius. To evaluate $\left\langle \vec{S}_e \bullet \vec{S}_p \right\rangle$ we define a total angular momentum vector $\vec{S} = \vec{S}_e + \vec{S}_p$, just as we did before to evaluate the spin-orbit perturbation. Note that $\vec{L} = 0$ here by assumption. This yields

$$\langle \vec{S}_e \bullet \vec{S}_p \rangle = \frac{1}{2}\left(\langle S^2 \rangle - \langle S_e^2 \rangle - \langle S_p^2 \rangle\right).$$

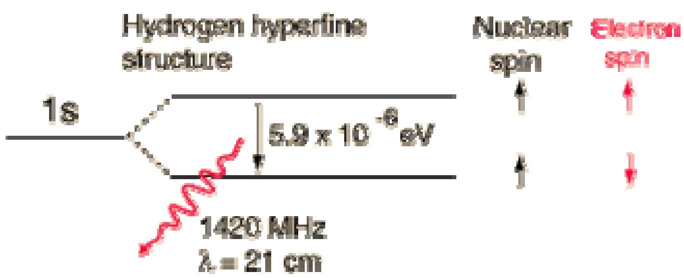
Note that the electron and proton are both spin-1/2 particles and so $\langle S_e^2 \rangle = \langle S_p^2 \rangle = 3\hbar^2/4$, and

$$\langle \vec{S}_e \bullet \vec{S}_p \rangle = \frac{1}{2}\left(\langle S^2 \rangle - \frac{3\hbar^2}{2}\right)$$

We treated the total spin of two spin-1/2 particles in the last lecture (Lecture 13) and found that two ladders of states are possible, that of $s=1$(the 3-state Triplet with $\langle S^2 \rangle = 2\hbar^2$) and $s=0$(the 1-state Singlet with $\langle S^2 \rangle = 0\hbar^2$). This yields two possible values for the first-order corrected energy:

$$E^1_{n,0,0} = \frac{\mu_0 g e^2}{3m_e m_p}\frac{\hbar^2}{\pi n^3 a^3}\begin{cases} 1/4 & \text{TRIPLET} \\ -3/4 & \text{SINGLET} \end{cases}$$

Consider the case of $n=1$, which is the ground state of Hydrogen (1s). This state is now split into two hyperfine-split states as shown in the diagram.



The energy splitting is only about 6 μeV, compared to the ground state binding energy of 13.6 eV. The upper state has a lifetime of about 10^{15} seconds, or about 10^8 years. When the atom makes a transition from the triplet state to the singlet state, it gives off radiation of frequency 1.420 GHz, with a wavelength of about 21 cm. This radiation can propagate through clouds of dust in the galaxy. From measurements of the Doppler shift of this radiation, the spiral structure of our galaxy was deduced. This transition photon was also used as the standard of length and time in the "post card" attached to the Pioneer 10 spacecraft.

Note that the picture of the orientation of the Nuclear spin and Electron spin in the above figure is somewhat deceiving. The actual states are described by the triplet and singlet spin wavefunctions (lecture 13), and can not be understood in terms of the "un-coupled" representation illustrated with the black and red arrows.