Extra Credit

The Schrödinger equation for the Macroscopic Quantum Wavefunction $\Psi(r,t)$ for a superconductor is

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m^*} \left( -i\hbar \vec{\nabla} - q^* \vec{A} \right)^2 \Psi + q^* \phi \Psi,$$

where $\vec{A}$ is the vector potential, $\phi$ is the scalar potential, $m^*$ and $q^*$ are the effective mass and charge of a Cooper pair. The macroscopic quantum wavefunction is interpreted as

$$\Psi(r,t) = \sqrt{n^*(r,t)} e^{i\theta(r,t)},$$

where $n^*(r,t)$ is the local number density and $\theta(r,t)$ is the space and time-dependent phase.

a) Under the assumption that the number density $n^*(\vec{r},t) = |\Psi(\vec{r},t)|^2$ is constant in space and time, derive the energy-phase relationship:

$$-\hbar \frac{\partial \theta}{\partial t} = \frac{1}{2n^*} \Lambda \vec{J}_s^2 + q^* \phi$$

from the real part of the macroscopic quantum Schrödinger equation. Interpret this equation physically. Here the supercurrent density $\vec{J}_s = \frac{1}{\Lambda} \left( \frac{\hbar}{q^*} \vec{\nabla} \theta - \vec{A} \right)$ and

$$\Lambda = \frac{m^*}{n^* (q^*)^2}.$$

b) Now assume that $n^*(\vec{r},t)$ is NOT constant in either space or time. Show that the imaginary part of the macroscopic Schrödinger equation yields:

$$\frac{\partial n^*}{\partial t} = -\vec{\nabla} \cdot (n^* \vec{v}_s)$$

Interpret this result physically (it may help to multiply both sides by $q^*$). Note that the superfluid velocity is given by

$$\vec{v}_s = \frac{\hbar}{m^*} \vec{\nabla} \theta - \frac{q^*}{m^*} \vec{A}.$$