March 28, 2008

Open Quantum Systems.

The H.O. Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

eq. of motion

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 q.$$

How do we introduce damping?

add a velocity dependent force $-\gamma p$

then the eq. of motion change to:

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -\gamma p - m\omega^2 q.$$

These can be recast as:

$\ddot{q} + \gamma \dot{q} + \omega^2 q = 0$ which is

the familiar damped harmonic oscillator.

One can just change this into the QM version as we do without damping.

use the operators $\hat{q}$ and $\hat{p}$

the Hamiltonian and the commutation relation: $[\hat{q}, \hat{p}] = i\hbar$

you can see now that we get the same linear equations of motion for the operators. the classical solutions still hold.

$$\dot{\hat{q}} = \hat{p}/m \qquad \dot{\hat{p}} = -\gamma\hat{p} - m\omega^2\hat{q}$$

now the expectation values of $\hat{q}$ and $\hat{p}$ are damped in the same way as the classical variables. Good!

→ Heisenberg picture, the operators have the time evolution

Let us now calculate the evolution of the commutator

$$\frac{d}{dt}[\hat{q},\hat{p}] = \frac{d}{dt}(\hat{q}\hat{p} - \hat{p}\hat{q})$$

$$= \dot{\hat{q}}\hat{p} + \hat{q}\dot{\hat{p}} - \dot{\hat{p}}\hat{q} - \hat{p}\dot{\hat{q}}$$

use the eq. of motion for $\dot{\hat{q}}$ and $\dot{\hat{p}}$

$$\frac{d}{dt}[\hat{q},\hat{p}] = \cancel{\frac{\hat{p}}{m}\hat{p}} - \gamma\hat{q}\hat{p} - \cancel{m\omega^2\hat{q}\hat{q}} + \cancel{m\omega^2\hat{q}\hat{q}}$$

$$+ \gamma\hat{p}\hat{q} - \cancel{\frac{\hat{p}\hat{p}}{m}}$$

$$= -\gamma(\hat{q}\hat{p} - \hat{p}\hat{q})$$

$$= -\gamma[\hat{q},\hat{p}]$$

This is a very simple equation to solve:

$$[\hat{q}(t), \hat{p}(t)] = e^{-\gamma t}[\hat{q}(0), \hat{p}(0)]$$

$$= e^{-\gamma t} i\hbar$$

$$\uparrow$$

this implies

$$\Delta q \, \Delta p \geq \frac{1}{2} \hbar e^{-\gamma t}$$

<u>wrong!</u>

We have to be careful.

Laser action <u>needs</u> dissipation

We have to open the door if we want to know what is happening inside!

Information <u>can escape</u> and noise or other non-wanted information can enter.

It seems that what we did is fine when dissipation is not present

Q.M gives us energy levels exceedingly well. The eigenstates of the H.O are fine.

What is really the problem!

$$\frac{dq}{dt} = \frac{p}{m} \qquad \frac{dp}{dt} = -m\omega^2 q$$

make a T transformation. $t \to -t$

$p \to -p$

$q \to q$.

$$-\frac{dq}{dt} = -\frac{p}{m} \qquad (-)(-)\frac{dp}{dt} = -m\omega^2 q$$

Same as before

T reversal invariant.

now let us look at

$$\frac{dp}{dt} = -\gamma p - m\omega^2 q$$

T

$$(-)(-)\frac{dp}{dt} = -(-)\gamma p - m\omega^2 q$$

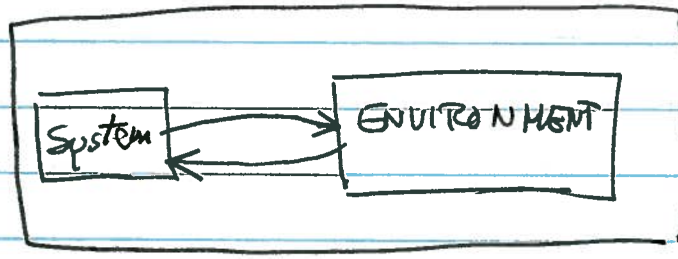
$$\frac{dp}{dt} = +\gamma p - m\omega^2 q$$

T is broken!

The origin of the irreversibility is tricky!

How is an oscillator damped?

It losses its information through interactions with a large complex system



if I open the door and the environment is some large system in thermal equilibrium it will exert a fluctuating force $F(t)$ on an oscillator coupled to it.

it will soak up the energy of the oscillator.

the familiar equation

$$\ddot{q} + \gamma \dot{q} + \omega^2 q = 0$$

becomes now a stochastic equation.

$$\ddot{q} + \gamma \dot{q} + \omega^2 q = \frac{F(t)}{m}$$

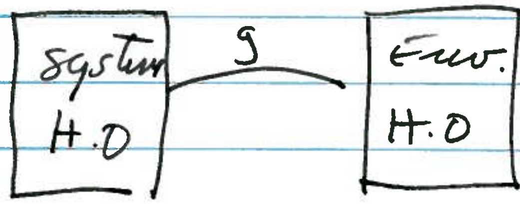
The added noise source can not be overlooked (for example in electrical circuits).

Damping is the coupling of the damped system to the environment.

Does this solve the commutator problem?

The interaction of one H.O coupled to a second H. O.

Let us now consider



Hamiltonian → resonant in the rotating wave approximation.

$$H = \underbrace{\hbar\omega\, \hat{a}^{+} \hat{a}}_{\text{energy of } \cancel{\text{syst}} \text{ oscillator}} + \underbrace{\hbar\omega\, \hat{b}^{+} \hat{b}}_{\text{energy of environment}} + \hbar g \left(\overset{\text{create}}{\hat{a}^{+}} \overset{\text{destroy}}{\hat{b}} + \underset{\text{destroy}}{\hat{a}} \underset{\text{create}}{\hat{b}} \right.$$

it is Hermitian.

remember

$$[\hat{a}, \hat{a}^{+}] = 1 \qquad [\hat{b}, \hat{b}^{+}] = 1$$

we have dropped the $\hbar\omega \frac{1}{2}$. (zero point energy)

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{q} + i\hat{p})$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{q} - i\hat{p})$$

what happened to terms like ab and a^+b^+
they are oscillating very fast and they do not conserve energy!

Now the commutator in the naive approach

$$[\hat{a}, \hat{a}^+] = e^{-\gamma t}$$

But now what do the solutions to the real coupled Hamiltonian give:

$$\hat{a}(t) = e^{-i\omega t}\left[\underline{\hat{a}(0)}\cos gt - \underline{i\hat{b}(0)}\sin gt\right]$$

MIXED.

$$\hat{b}(t) = e^{-i\omega t}\left[\hat{b}(0)\cos gt - i\hat{a}(0)\sin gt\right]$$

such that

$$[\hat{a}(t), \hat{a}^+(t)] = [\hat{a}(0), \hat{a}^+(0)]\cos^2 gt +$$

$$[\hat{b}(0), \hat{b}^+(0)]\sin^2 gt$$

$$= 1$$

now the commutator is preserved.
because of the presence of $\underline{\underline{\hat{b}}}$ mixed with the solution for $\hat{a}(t)$.

If you open the ~~door~~ in Q.M. information can get out. The laser lases! but fluctuations come in.

Dissipation mixes the environmental operators with those of the system

Why is it not reversible?

The system is small, the environment very large.

infinite

it would take an infinite amount of time to return to the system.

Are there other approaches possible?

Make $F(t) \to \hat{F}(t)$ an operator then that contributes to $\hat{p}(t)$ and $\hat{q}(t)$ introducing thermal fluctuations, ~~in~~ and helping preserve the commutation relations.

Whenever we measure a system in Q.M. we open the door → back action.

This is not seen as affecting the spectrum, of H or U. The energy levels of a H.O.

But if you are looking at quantum dynamics. just as in Q.O. The laser, cavity QED and the such. watch out!

Please note that this is not the full picture of Quantum Measurement theory.