

## Physics 401 Homework 9---Due November 18

1. Prove the following:
  - a. If  $\hat{T}|\psi\rangle = |\phi\rangle$  then  $\langle\psi|\hat{T}^\dagger = \langle\phi|$
  - b.  $(\hat{T}\hat{S})^\dagger = (\hat{S}^\dagger\hat{T}^\dagger)$
  - c.  $(\hat{T}\hat{S})^{-1} = (\hat{S}^{-1}\hat{T}^{-1})$
  - d. Any Hermitian operator  $\hat{T}$  can be decomposed as  $\hat{T} = \hat{S} + i\hat{A}$  where  $\hat{S}$  is a symmetric operator whose matrix elements satisfy  $S_{ij} = S_{ji}$  and  $\hat{A}$  is an antisymmetric operator whose matrix elements satisfy  $A_{ij} = -A_{ji}$ . Prove this by explicit construction of  $\hat{S}$  and  $\hat{A}$ .
  - e. If two operators,  $\hat{H}_1$  and  $\hat{H}_2$  are each Hermitian then  $\hat{H} = \hat{H}_1 + \hat{H}_2$  is also Hermitian.
  - f. If two operators,  $\hat{U}_1$  and  $\hat{U}_2$  are each unitary then  $\hat{U} = \hat{U}_2\hat{U}_1$  is also unitary.
  - g. If  $\hat{H}$  is Hermitian and  $\hat{U}$  is unitary then  $\hat{H}' = \hat{U}^\dagger\hat{H}\hat{U}$  is also Hermitian.
  - h. If  $\hat{A}' = \hat{U}^\dagger\hat{A}\hat{U}$  and  $\hat{B}' = \hat{U}^\dagger\hat{B}\hat{U}$  then  $\hat{U}^\dagger\hat{A}'\hat{B}'\hat{U} = \hat{A}'\hat{B}'$ .
  
2. The purpose of this problem is to construct the matrix for the time evolution operator in the energy eigenbasis. This is the orthonormal basis of states which are eigenstates of the Hamiltonian,  $\hat{H}$ . The basis can be written as the set  $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle, \dots\}$  where  $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$ .
  - a. Consider an operator  $\hat{U}(t)$  whose matrix elements are given by  $U_{jk}(t) \equiv \langle\psi_j|\hat{U}(t)|\psi_k\rangle = \delta_{jk} e^{-iE_k t/\hbar}$ . The matrix  $\vec{U}(t)$  is therefore diagonal. Show explicitly that  $\vec{U}(t)$  is a unitary matrix:  $\vec{U}^\dagger(t)\vec{U}(t) = \vec{\mathbf{1}}$  where  $\vec{\mathbf{1}}$  is the identity matrix. This implies that  $\hat{U}(t)$  is a unitary operator.
  - b. Show that the operator  $\hat{U}(t)$  from part a. can be written as 
$$\hat{U}(t) = \sum_n |\psi_n\rangle\langle\psi_n| e^{-iE_n t/\hbar}$$
 where the sum on  $n$  is over all of the eigenstates. To show this you must show that all matrix elements of  $\hat{U}(t)$  are the ones given in part a.
  - c. Using the form in a., show explicitly that  $\hat{U}(t)$  is the time evolution operator. That is show that it satisfies  $\hat{H}\hat{U}(t) = i\hbar \frac{d\hat{U}(t)}{dt}$  with  $\hat{U}(0) = \hat{\mathbf{1}}$  where  $\hat{\mathbf{1}}$  is the identity operator.