

1. a. Evaluate the commutators  $[x, L_z]$ ,  $[z, L_z]$ ,  $[p_x, L_z]$ ,  $[p_z, L_z]$ . Here  $(x, y, z)$  and  $(p_x, p_y, p_z)$  are position and momentum operators, and  $\vec{L}$  is angular momentum, for example  $L_z = xp_y - yp_x$ . By “evaluating” I mean reducing the expression to a single position or momentum operator.

The only operator in  $L_z$  that does not commute with  $x$  is  $p_x$ , thus  $[x, L_z] = -y[x, p_x] = \hbar y/i$ . Similarly  $[z, L_z] = 0$ ,  $[p_x, L_z] = [p_x, x]p_y = \hbar p_y/i$ ,  $[p_z, L_z] = 0$ .

b. By cyclic exchange of indices the four commutators of part a. allow you to write down any commutator of the form  $[\vec{r}, \vec{L}]$ ,  $[\vec{p}, \vec{L}]$ . For example, evaluate  $[x, L_x]$  and  $[y, L_x]$ .

You find  $[x, L_x] = 0$ ,  $[y, L_x] = -x\hbar/i$

c. Prove: If the operators  $A$  and  $B$  are simultaneously measurable, then  $[A, B] = 0$ . Here “simultaneously measurable” means that there is a complete set of functions that are eigenfunctions of  $A$  as well as of  $B$  (with possibly different eigenvalues  $a_i, b_j$ ).

Let  $\psi_{ij}$  be such a set of eigenfunctions,  $A\psi_{ij} = a_i\psi_{ij}$ ,  $B\psi_{ij} = b_j\psi_{ij}$ . Then  $AB\psi_{ij} = a_ib_j\psi_{ij} = BA\psi_{ij}$ . Since the  $\psi_{ij}$  are a complete set, any  $\psi$  can be expanded,  $\psi = \sum c_{ij}\psi_{ij}$  and we have  $AB\psi = \sum c_{ij}a_ib_j\psi_{ij} = BA\psi$ .

d. If  $A$  and  $B$  are simultaneously measurable, and  $B$  and  $C$  are simultaneously measurable, it may seem reasonable that  $A$  and  $C$  should be simultaneously measurable. That is,  $[A, B] = 0$  and  $[B, C] = 0$  would imply  $[A, C] = 0$ .

Is this correct? If it is, prove it. If not, give a counterexample.

This is not correct, for example  $[x, y] = 0$  and  $[y, p_x] = 0$  but  $[x, p_x] \neq 0$ .

2. **This problem will be the first problem on Assignment 1 of Phys402.** It is very similar to Griffith’s problem 2.13. Frequent mistakes on the final were

- failure to complex conjugate in  $|\Psi|^2 = \Psi^*\Psi$
- believing  $\Psi_n^*\Psi_m$  vanishes for two orthogonal functions, whereas only  $\int \Psi_n^*\Psi_m dx = 0$ .

A one-dimensional simple harmonic oscillator is governed by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad (*)$$

In classical physics this oscillator oscillates at only one frequency  $\omega$ . What is the situation in quantum mechanics? Let  $\psi_n$  be the time-independent  $n^{\text{th}}$  excited state of the oscillator. The corresponding time-dependent  $\Psi_n$  is a stationary state, so the probability density  $\Psi_n^*\Psi_n$  is of course constant and does not oscillate. Now suppose it starts out in the superposition state

$$\Psi(x, 0) = \frac{1}{\sqrt{2}}(\psi_n(x) + \psi_m(x)) \quad \text{at } t = 0.$$

a. Construct  $\Psi(x, t)$  and  $|\Psi(x, t)|^2$ .

b. With what frequency (or frequencies) does its probability density  $|\Psi(x, t)|^2$  oscillate in time? Express your answer in terms of the  $\omega$  in  $H$  of (\*), and  $n$  and  $m$ .

c. With what frequency (or frequencies) will the expectation value  $\langle x \rangle$  oscillate in time? Note that for many values of  $n$  and  $m$ ,  $\langle x \rangle$  vanishes for all times, hence it does not have an oscillation frequency. Do not include such “zero-amplitude” frequencies in your answer. That is, find a condition on  $n$  and  $m$  that guarantees a non-zero amplitude of oscillation.

Here  $n \neq m$  or  $n = m + 1(\text{mod}2)$  (that is,  $n$  and  $m$  must have different parity) is not enough as the condition. Express  $x$  in terms of raising and lowering operators.

3. This question concerns the situation of the take-home exam, where the potential in a hydrogen atom is supplemented by another potential  $\lambda/r^2$ , with  $\lambda$  a positive constant. A student writes for the energy

$$E_n = - \left[ \frac{m}{2\hbar^2} \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} + \frac{\lambda}{r^2}$$

where  $r$  is the radial coordinate. The first term is the energy of the pure H atom.

a. Without going into detail one can see that this cannot be a correct formula for an energy eigenvalue. What is the problem, and why?

The expression depends on  $r$ , but eigenvalues are constants and should not depend on the position variables of the Schrödinger representation.

b. When confronted with this problem, the student replies “I meant to write  $\langle \lambda/r^2 \rangle$ , where the expectation value is to be taken for the  $n^{\text{th}}$  state of pure hydrogen.” Does the so-changed formula give the exact energy of the supplemented H atom? Or a reasonable approximation for small  $\lambda$ ? Or is it still all wrong? Give reasons for your answer!

The linear dependence on  $\lambda$  is suspicious, for large  $\lambda$  the energy would be positive and quantized. So the expression cannot be exact, but it is not unreasonable for small  $\lambda$ . Here the fact that it is exact for  $\lambda = 0$  is however not sufficient reason, because that would be true no matter what the coefficient of  $\lambda$  is (instead of  $\langle 1/r^2 \rangle$ ). But note that the student’s suggestion amounts to  $\langle H \rangle$  evaluated in a pure hydrogen state. For small  $\lambda$  we expect the pure hydrogen state to be not far different from the “supplemented” hydrogen state. For the latter, exact state,  $\langle H \rangle$  of course does equal the exact energy.

c. Can one write the exact energy eigenvalue of any Hamiltonian as an expectation value? How can one get the different eigenvalues from such an expression? Why is this way to get the energy eigenvalues not a useful alternative to solving the time-independent Schrödinger equation?

Yes, as noted in part b. – and actually for any operator – if  $|n\rangle$  is an exact, normalized solution of  $H|n\rangle = E_n|n\rangle$ , then  $\langle n|H|n\rangle = E_n$ .

You get the different eigenvalues by evaluating the expectation value in the different eigenstates  $|n\rangle$ . For this you need to exact eigenstates, and usually the only way to get them is to solve the time-independent Schrödinger equation – so it is not a real alternative.

**4. This problem will be the first problem on Assignment 1 of Phys402.** It is similar to Griffiths’ problem 3.23. A basic difficulty on the exam was the meaning of a notation like  $|1\rangle\langle 2|$ . This is an operator, defined by what it does to any state  $|s\rangle$ :  $(|1\rangle\langle 2|)|s\rangle = |1\rangle(\langle 2|s\rangle)$ , that is, it gives state 1 multiplied by the scalar product of state 2 and state  $s$ . As a matrix in the basis  $|1\rangle, |2\rangle$  it takes the form  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

A quantum mechanical system can exist in 3 states only,  $|0\rangle, |1\rangle$  and  $|2\rangle$ . They are orthonormal  $\langle i|j\rangle = \delta_{ij}$ , and the Hamiltonian is

$$H = E(|1\rangle\langle 1| + |2\rangle\langle 2|) + \lambda(|0\rangle\langle 1+2| + |1+2\rangle\langle 0|).$$

Here  $|1+2\rangle$  means  $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ .  $\lambda$  is a real constant.

If there is no interaction between the states, that is if  $\lambda = 0$ , then  $|0\rangle$  is the ground or vacuum state, the other two are degenerate excited states of energy  $E$ . Because of the interaction term with  $\lambda$  the states  $|i\rangle$  are not stationary.

a. Evaluate  $H|1-2\rangle = \frac{1}{\sqrt{2}}H(|1\rangle - |2\rangle)$  and thus show that  $|1-2\rangle$  is stationary (that is, it’s an eigenstate of  $H$ ). What is its energy?

b. Find the expectation value of the energy in the state  $|1+2\rangle$

c. Write  $H$  in matrix form and find the energies of the other two stationary states of this Hamiltonian.



Formulas

Time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = H\psi(x)$$

Time-independent Schrödinger equation	$H\psi(x) = E\psi(x)$
Hamiltonian operator	$H = \frac{p^2}{2m} + V(x)$
Momentum operator	$p = \frac{\hbar}{i} \frac{\partial}{\partial x}$
Expansion	$ \psi\rangle = \sum  e_n\rangle \langle e_n   \psi \rangle$
Commutator	$[p, x] = \frac{\hbar}{i}$
Expectation value	$\langle Q \rangle = \langle \psi   Q   \psi \rangle$
Matrix elements	$Q_{mn} = \langle e_m   Q   e_n \rangle$
Allowed energies of H atom	$E_n = - \left[ \frac{m}{2\hbar^2} \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2}$