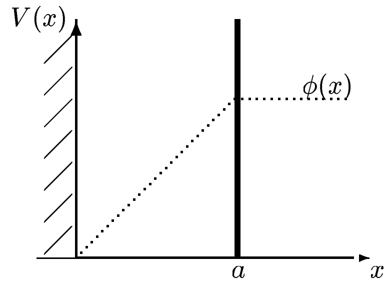
Physics 401 | Hour Exam Solution | March 12, 2008

Use exam booklet. Write legibly and draw clear, labeled diagrams. Don't forget to sign the pledge; also put a legible version of your name in the space provided.
Each part of each problem counts 10 points. 10 points for complying with the above instructions.



1. In this problem we have the usual particle of mass m and the potential

$$V(x) = \begin{cases} \infty & \text{for} \quad x \le 0 \\ -\alpha\delta(x-a) & \text{for} \quad x > 0 \end{cases}$$

which is shown on the right (for $\alpha < 0$, to save space).

(a) If you had to integrate the time-independent Schrödinger equation for this type of potential that is *attractive* ($\alpha > 0$), where would you start the integration, and what "initial" values for ψ and $d\psi/dx$ would you chose at the starting point?
It is best to start the integration at a finite value of x where we know the boundary conditions, so $x = 0$ is the favored choice of starting point. (We also know boundary conditions at $x = \infty$, but to make that useful you'd have to change variables, say to $\arctan(x/a)$, to put this at a finite value.)
Since $V(0) = \infty$ we must have $\psi(0) = 0$. Then to get a $\psi(x) \neq 0$ we need $\psi'(0) \neq 0$, for example $\psi'(0) = 1$. (Any other constant would do equally well, it only changes $\psi(x)$ by an overall factor. If a normalized wave function is needed, the normalization factor must be determined *after* a numerical, non-normalized $\psi(x)$ has been found).

(b) You have guessed a value of the energy E. The numerical integration coped correctly with the delta function in the potential, and gave you a numerical $\psi(x)$ that increases monotonically with x. For the next "wag of the dog," should you increase or decrease E?
$\psi(x)$ can't keep increasing, it has to get to zero at $x = \infty$, so it has to bend down, hence it must be given greater curvature *toward* the axis. The time-independent Schrödinger equation implies that to achieve this you must **increase** the energy.

(c) Actually, numerical integration is not needed. By matching appropriate solutions in the two regions, find a transcendental (but not differential) equation for the ground state energy, in terms of m, $\hbar$, a, α. Also sketch a typical wave function.
For a bound state the energy will be negative, hence outside the delta function the wave function will be a combination of real exponentials. To satisfy the boundary condition at $x = 0$ we choose $\sinh(\kappa x)$ on the left of $x = a$, and to get $\psi(\infty) = 0$ we choose $\exp(-\kappa x)$ on the right of $x = a$:

$$\psi(x) = \begin{cases} \sinh\kappa x & \text{for} \quad 0 < x < a \\ Ae^{-\kappa x} & \text{for} \quad x > a \end{cases}$$

where $\kappa = \sqrt{2m(-E)}/\hbar$. Matching values at $x = 0$ gives us

$$\sinh\kappa a = Ae^{-\kappa a}.$$

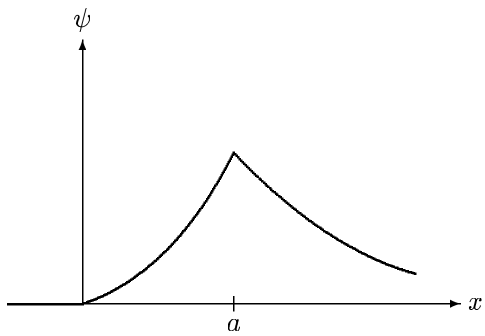
Due to the δ-function the slope changes by $-\frac{2m\alpha}{\hbar^2}\psi(a)$ at $x = a$:

$$\kappa\cosh\kappa a = A\kappa e^{-\kappa a} - \frac{2m\alpha}{\hbar^2}Ae^{-\kappa a} = \kappa\sinh\kappa a\left(1 - \frac{2m\alpha}{\hbar^2\kappa}\right)$$

or

$$\coth\kappa a = 1 - \frac{2m\alpha}{\hbar^2\kappa}.$$

This, together with $E = -\frac{\kappa^2\hbar^2}{2m}$ is the required transcendental equation for E.



(d) For this potential and a particular "critical" $\alpha_c > 0$, an exact wave function is

$$\phi(x) = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ x & \text{for} \quad 0 < x < a \\ a & \text{for} \quad x \geq a \end{cases}$$

as shown by the dotted line in the figure above. What is the energy E for this state?
Since on the left and on the right of the δ function, $d^2\psi/dx^2 = 0$ we must have $E = 0$.

(e) Find α_c for the case in (d), in terms of m, $\hbar$, a. Is there a bound state if $\alpha < \alpha_c$?
The change in slope is -1, this must equal $-\frac{2m\alpha_c}{\hbar^2}\psi(a) = -\frac{2m\alpha_c a}{\hbar^2}$ hence

$$\alpha_c = \frac{\hbar^2}{2ma}.$$

If $\alpha < \alpha_c$ the change in slope is less than in (d), so ψ increases monotonically if $E = 0$, as in part (b). To make ψ "bend down" we would have to increase E, which would be > 0, hence no longer a bound state. So there is no bound state for $\alpha < \alpha_c$.

2. We have a potential of "even parity", that is, $V(x)$ is an even function, $V(x) = V(-x)$, and stationary bound states $\psi_n(x)$ in this potential. These energy eigenstates are non-degenerate.

(a) Prove: if $\psi(x)$ is a solution of the time-independent Schrödinger equation, then $\psi(-x)$ also solves this equation. From this, argue that $\psi_n(x)$ must have definite parity, that is, either $\psi_n(x) = \psi_n(-x)$ or (for a different n) $\psi_n(x) = -\psi_n(-x)$
It does not matter what we call the independent variable in the Schrödinger equation, so call it y. Then $\psi(y)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(y)}{dy^2} + V(y)\psi(y) = E\psi(y)$$

Now let $y = -x$, note $d^2/dy^2 = d^2/dx^2$ to find

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(x)\psi(-x) = E\psi(-x),$$

which is the Schrödinger equation for $\psi(-x)$.
Therefore $\psi(x) + \psi(-x)$ and $\psi(x) - \psi(-x)$ must also solve the Schrödinger equation. The former is even, the latter, odd. They are clearly independent. But the one-dimensional equation has unique eigenstates, so one of them must vanish: ψ is either even or odd.

(b) Suppose $\psi(x)$ is some even wave function, not necessarily one of the energy eigenstates – in other words, a state that can have non-trivial time dependence. Show that parity is conserved in time, that is "once even (odd) always even (odd)." What is $\langle x(t) \rangle$ and $\langle p(t) \rangle$ for these states?
If ψ is even (odd), then $H\psi$ is also even (odd) (as we saw above, d^2/dx^2 does not change parity, neither does the even $V(x)$.) Since $i\hbar d\Psi/dt = H\psi$ at $t = 0$, $d\Psi/dt|_{t=0}$ will also be even/odd, and similarly all higher derivatives; thus Ψ stays even (odd).
Alternatively: $\Psi(x,t) = \sum c_n \psi_n(x) e^{-iE_n t/\hbar}$. If $\Psi(x,0)$ is even (odd), its expansion will contain only

even (odd) ψ_n (since even is orthogonal to odd). The factor $e^{-iE_n t/\hbar}$ is even (does not depend on x at all), hence does not change the parity for any value of t.
$|\Psi|^2$ is even but x and p are odd, hence $\langle x(t)\rangle = 0 = \langle p(t)\rangle$.

(c) An example of a state that is even and has non-trivial time dependence, is the normalized superposition of the $n = 1$ and 3 states in the infinite square well of width a, centered about $x = 0$. At $t = 0$ this is

$$\psi(x) = \sqrt{\frac{1}{a}}\left(\cos\frac{\pi x}{a} + \cos\frac{3\pi x}{a}\right)$$

Show that the probability density $|\Psi(x,t)|^2$ is not constant in time, for example by evaluating it at $x = 0$.
The energies corresponding to $n = 1$ and 3 are $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ and $E_3 = 9E_1$, therefore

$$|\Psi(0,t)|^2 = \frac{1}{a}\left|e^{-iE_1 t/\hbar} + e^{-9iE_1 t/\hbar}\right|^2 = \frac{1}{a}[2 + 2\cos(8E_1 t/\hbar)].$$

(d) In the state of part (c) there must be some quantities (operators) whose expectation value is time-dependent. Mention two such expectation values.
To get something $\neq 0$ – see (b) – we need an even function. $\langle x^2(t)\rangle$ is the obvious choice, but $\langle p^2(t)\rangle$ will *not* be time-dependent: it is essentially the kinetic energy, which is constant for the infinite square well. But $\langle x^2(t)p^2(t)\rangle$ would work (though hardly different from $\langle x^2(t)\rangle$), as would $\langle x^4(t)\rangle$.