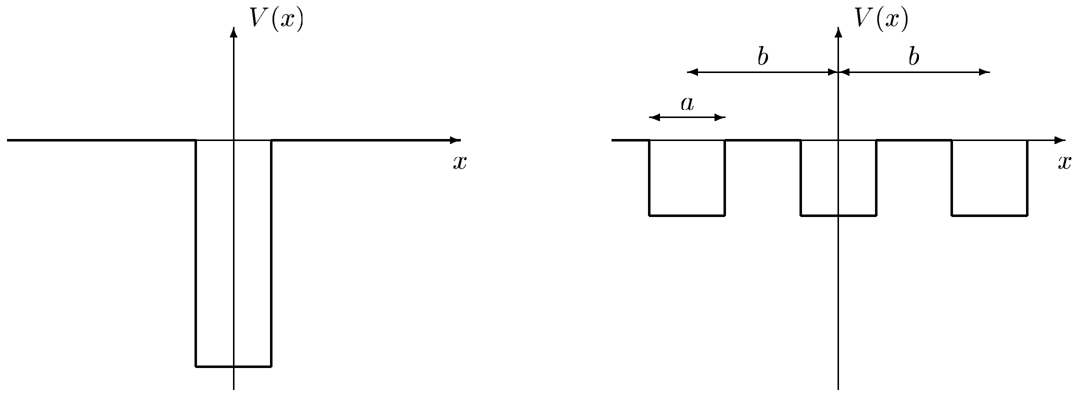
Physics 401 **Midterm Exam Solutions** March 7, 2007

1. Consider the "triple square well" consisting of three square wells of width a with centers separated by a distance b. When $b = 0$ they coincide and make a well three times as deep, when $b \gg a$ they are three well-separated wells.

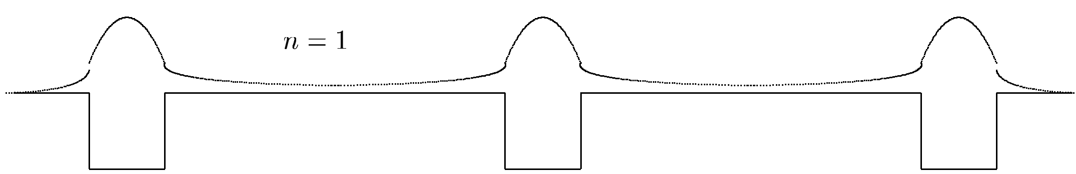


Suppose the depth and width are large enough so that at least 3 bound states exist in the wells, and that their energies approximately follow the pattern of the infinite square well.

(a) sketch the ground state and first two excited state wave functions for the case $b = 0$ and for $b \gg a$. Make sure your sketches show the proper symmetry (even or odd) about $x = 0$.

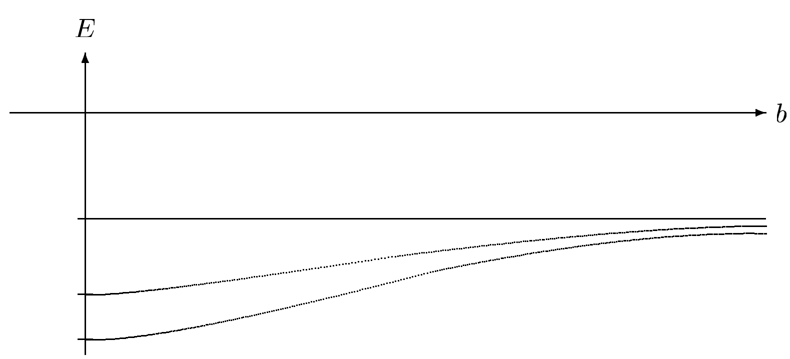
For $b = 0$: similar to figure 2.2 but with some small tails going to infinity

For $b \gg a$ (potential and wavefunction on same graph:



The next (first) excited state has a similar hump on one side, the negative of that on the other, and a zero in the middle (it is very small everywhere in the central potential. The second excited state has two positive humps on the ends and a negative one in the middle (or vice versa), crossing zero in the spaces between the potential wells.

(b) The energies of the states are continuous, even functions of b (because $+b$ and $-b$ corresponds to the same potential). Sketch $E_1(b)$, $E_2(b)$, and $E_3(b)$ on the same graph.



(Whether the excited states rise or fall depends on the details of the potential. E_3 was drawn straight only because that's easier than drawing curves)

2. At $t = 0$ the wavefunction for a particle of mass m in a one-dimensional infinite square well of size a is given by

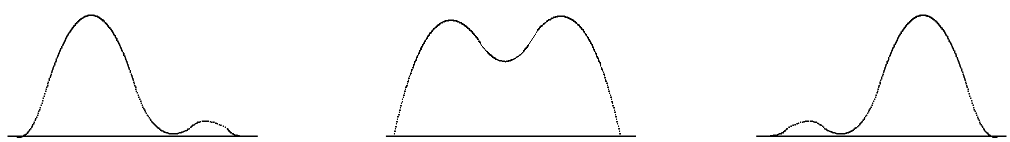
$$\Psi(x,0) = \sqrt{\frac{1}{a}}\left(\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)\right) \qquad 0 \le x \le a$$

(a) What is the time dependent solution of the Schrödinger equation, $\Psi(x,t)$?

$$\Psi(x,t) = \sqrt{\frac{1}{a}} e^{-i\frac{E_1}{\hbar}t}\left(\sin\left(\frac{\pi x}{a}\right) + e^{-i\frac{3E_1}{\hbar}t}\sin\left(\frac{2\pi x}{a}\right)\right)$$

(b) Make a rough plot of $|\Psi(x,t)|^2$ at $t = 0$, $ma^2/3\pi\hbar$, $2ma^2/3\pi\hbar$.

At those times the relative phase factor $e^{-i\frac{3E_1}{\hbar}t}$ is 0, -i and -1 respectively. Hence we plot $\left(\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)\right)^2$, $\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)$, and $\left(\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{2\pi x}{a}\right)\right)^2$
which look like this:



3. A position measurement is made on the one-dimensional Simple Harmonic Oscillator (mass m, frequency ω, giving $x = 0$.

(a) What is the wavefunction immediately after the measurement?

It is a delta function centered at x = 0, that is, $\psi(x) = \delta(x)$

(b) In a subsequent measurement of energy, what are the probabilities of finding the oscillator in its first and second excited states, relative to that of the ground state?[1]
Possibly helpful remark: for the SHO we have

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}, \quad \psi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}}(2\xi^2 - 1)e^{-\xi^2/2} \quad \text{where } \xi^2 = \frac{m\omega x^2}{\hbar}$$

We have $c_0 = \int \psi_n(x)\delta(x)dx = \psi_n(0)$ So $c_1/c_0 = \psi_1(0)/\psi_0(0) = 0$ because you know (though not given in the problem) that $\psi_1(x)$ is an *odd* function.
Also $c_2/c_0 = \psi_2(0)/\psi_0(0) = 1/\sqrt{2}$. So the relative probabilities are zero for the first excited state and $\frac{1}{2}$ for the second excited state.

4. Problem 3 was too easy, so do it again: Think of ψ as expanded in SHO eigenfunctions ($|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + ...$), express the condition on ψ, that an x-measurement gave $x = 0$, using the position operator as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-),$$

Possibly helpful: $a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $a_-|n\rangle = \sqrt{n}\,|n-1\rangle$

The equation $\hat{x}\psi \propto (a_+ + a_-)|\psi\rangle = 0$ becomes

$$c_0|1\rangle + c_1(\sqrt{2}|2\rangle + |0\rangle) + c_2(\sqrt{3}|3\rangle + \sqrt{2}|1\rangle + c_3(\sqrt{4}|4\rangle + \sqrt{3}|2\rangle) + ... = 0$$

[1] That is, in terms of the expansion of ψ as in the next problem, you want $|c_1|^2/|c_0|^2$ and $|c_2|^2/|c_0|^2$ You can only compute relative probabilities because the answer to (a) is not normalized the same way as the SHO states.

The coefficient of each $|n\rangle$ must vanish separately, so we get

$$\text{for } |0\rangle : c_1 = 0 \qquad \text{for } |1\rangle : c_0 + \sqrt{2}c_1 = 0 \qquad \text{for } |2\rangle : \sqrt{2}c_2 + \sqrt{3}c_3 = 0$$

Thus $c_1 = 0$, $|c_2/c_0|^2 = \frac{1}{2}$, $c_3 = 0$ $c_4 = \sqrt{3/8}$.... Although the c's converge to zero, the sum of their squares diverges (slowly) – the δ-function state cannot be normalized.

5. Let A be a hermetian operator. Assume that the inverse A^{-1} of A exists.[2]

(a) Show that the eigenvalues of A^{-1} are the inverses of those of A.

Let $A\psi_n = a_n\psi_n$. Let A^{-1} act on this: $A^{-1}A\psi_n = \psi_n = a_nA^{-1}\psi_n$. Solve for $A^{-1}\psi_n$: $A^{-1}\psi_n = (1/a_n)\psi_n$.

(b) Show that the eigenvalues of A^n are the n^{th} power of those of A, for n positive or negative.

$A^n\psi_k = A^{n-1}(A\psi_k) = A^{n-1}a_k\psi_k = a_kA^{n-2}(A\psi_k) = a_k^2A^{n-3}(A\psi_k) = \text{etc} = a_k^n\psi_k$. Similarly for A^{-n}, or use method of part (a).

6. Sometimes a perfectly reasonable (for example, algebraic) expression Q, in terms of the basic dynamic variables x and p, is not hermetian when x and p in Q are replaced by the corresponding operators.

(a) Show that this is the case for xp, that is, show that $\hat{x}\hat{p}$ is *not* hermetian and find its hermetian adjoint.

$\langle f|xp\,f\rangle = \langle f|x(pf)\rangle = \langle xf|pf\rangle = \langle pxf|f\rangle$ so $(xp)^\dagger = px$. (Other answers possible, using $[x,p] = i\hbar$.)

(b) On the other hand, show that the *sum* of any two hermetian operators, as well as the *square* of any hermetian operator, is hermetian.

Let A, B be hermetian. Then $\langle f|(A+B)f\rangle = \langle f|AF\rangle + \langle f|Bf\rangle = \langle Af|f\rangle + \langle Bf|f\rangle = \langle (A+B)f|f\rangle$ since $\langle\ |\ \rangle$ is linear in either entry.

(c) Classically we have $xp = \frac{1}{2}[(x+p)^2 - x^2 - p^2]$. This should be hermetian when we replace the RHS by the corresponding operator expression. What is this hermetian operator? How is it related to $\hat{x}\hat{p}$ (and its adjoint)?

When the order of operators is important we must write $(x+p)^2 = x^2 + xp + px + p^2$, so that

$$\frac{1}{2}[(x+p)^2 - x^2 - p^2] = \frac{1}{2}(xp + px) = \frac{1}{2}(xp + (xp)^\dagger).$$

(d) This can be generalized to any operator $\hat{Q}$: Show that its "hermetian part", $\frac{1}{2}(\hat{Q}^\dagger + \hat{Q})$ is hermetian.

Note that $\langle f|\hat{Q}f\rangle = \langle \hat{Q}^\dagger f|f\rangle = (\langle f|\hat{Q}^\dagger f\rangle)^* = (\langle \hat{Q}^{\dagger\dagger} f|f\rangle)^* = \langle f|\hat{Q}^{\dagger\dagger} f\rangle$. Therefore $\hat{Q}^{\dagger\dagger} = \hat{Q}$ and $\frac{1}{2}(\hat{Q}^\dagger + \hat{Q})^\dagger = \frac{1}{2}(\hat{Q} + \hat{Q}^\dagger)$

7. x and p are the usual operators for position and momentum.

(a) The commutator $[x^n, p]$ can be simplified so it becomes proportional to a power of the x operator only. Find this equivalent expression.

$$[x^n, p]f = x^n\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{dx^nf}{dx} = \frac{\hbar}{i}\left(x^n\frac{df}{dx} - nx^{n-1}f - x^n\frac{df}{dx}\right) = nx^{n-1}f$$

therefore $[x^n, p] = \frac{\hbar}{i}nx^{n-1}$.

(b) Similarly, $[p^n, x]$ is proportional to a power of p only. Derive that expression. (This is quite simple in the momentum representation.)

In the momentum representation, $\hat{p} = p\cdot$ and $\hat{x} = -\frac{\hbar}{i}\frac{d}{dp}$, so p and x are interchanged except for a minus sign. Hence $[p^n, x] = -\frac{\hbar}{i}np^{n-1}$

[2] this means that for any function g there exists a function f with $g = Af$; then $f = A^{-1}g$. Alternatively, $A^{-1}A = AA^{-1} = \mathbb{1} =$ unit operator.