QUANTUM PHYSICS I

MIDTERM 2

A. Step potential

A particle of mass m moves under the influence of a potential

$$V(x) = \begin{cases} V_0, \text{ for } x < 0 \\ 0 \text{ for } x > 0 \end{cases}, \tag{1}$$

where $V_0 > 0$. What is the probability that a particle moving from the left with energy $E > V_0$ will be reflected back to the left side?

Solution of the Schroedinger equation:

$$\begin{align} x > 0 : \; -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= E\psi(x) \rightarrow \psi(x) = Ce^{ikx} + De^{-ikx}, \tag{2} \\ x < 0 : \; -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= (E - V_0)\psi(x) \rightarrow \psi(x) = Ae^{ilx} + Be^{-ilx}, \tag{3} \\ & \tag{4} \end{align}$$

where $\hbar k = \sqrt{2mE}$ and $\hbar l = \sqrt{2m(E - V_0)}$. The choice $D = 0$ corresponds to no particle coming from the right. We now impose the continuity of $\psi(x)$ and its derivative at $x = 0$:

$$\begin{align} A + B &= C. \tag{5} \\ il(A - B) &= ikC. \tag{6} \\ & \tag{7} \end{align}$$

Dividing one equation above by the other we find

$$il\frac{A - B}{A + B} = ik \rightarrow l(1 - \frac{B}{A}) = k(1 + \frac{B}{A}) \rightarrow \frac{B}{A}(k + l) = l - k \rightarrow \frac{B}{A} = \frac{l - k}{k + l}. \tag{8}$$

The probability of reflection is given by the ratio of the probability current of the incoming wave and the reflected wave:

$$\text{probability of reflection} = \frac{|j_B|}{|j_A|} = \frac{\hbar l|B|^2/m}{\hbar l|A|^2/m} = \left|\frac{B}{A}\right|^2 = \left|\frac{\sqrt{2mE} - \sqrt{2m(E - V_0)}}{\sqrt{2mE} + \sqrt{2m(E - V_0)}}\right|^2. \tag{9}$$

B. Heavy-heavy mesons from the "Old Quantum Mechanics" perspective

There are elementary particles called "quarks" and "anti-quarks" which interact with each other with a force that is attractive, but independent of the distance between them. Suppose there are two of these quarks, one infinitely heavy and another of finite mass M (this is called a "heavy-light" meson). This system is very similar to a hydrogen atom, the only difference being the force law which is

$$V(r) = \alpha r, \tag{10}$$

instead of the Coulomb law. The goal of this problem is to determine the energy levels of this system using the "Old Quantum Mechanics" ideas used in the derivation of the Bohr model.

i) Using the Bohr quantization condition for the angular momentum and the force law above, determine the energy levels allowed.

$$\begin{aligned} L = Mvr \; &= \; n\hbar, \;\; n = 1, 2, \cdots && (11) \\ \alpha \; &= \; M\frac{v^2}{r}. && (12) \end{aligned}$$

From the equations above

$$\alpha = \frac{M}{r}\left(\frac{n\hbar}{Mr}\right)^2 \Rightarrow r = \left(\frac{n^2\hbar^2}{M\alpha}\right)^{1/3} \tag{13}$$

The total energy is given by

$$E = \frac{Mv^2}{2} + \alpha r = \frac{\alpha r}{2} + \alpha r = \frac{3\alpha r}{2} = \frac{3\alpha}{2}\left(\frac{n^2\hbar^2}{M\alpha}\right)^{1/3} = \frac{3}{2}\left(\frac{n^2\alpha^2\hbar^2}{M}\right)^{1/3} \tag{14}$$

ii) What are the wavelengths of electromagnetic radiation this system can emit? The energy of the photon emitted/absorbed correspond to the difference of the energy levels:

$$\lambda = \frac{\nu}{c} = \frac{E_n - E_m}{hc} = \frac{3}{2hc}\left(\frac{\alpha^2\hbar^2}{M}\right)^{1/3}\left(n^{2/3} - m^{2/3}\right) \tag{15}$$

C. Measurement & probabilities

A particle of mass m is in the ground state of an infinite square well potential with walls at $x = 0$ and $x = a$. Suddenly, the right wall of the potential moves to $x = L$.

i) What is the wave function of the particle before the wall moves.

The stationary states of the infinite square well can be found by solving the Schroedinger eq. with the appropriate boundary conditions:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \text{ and}\psi(0) = \psi(a) = 0 \Rightarrow \psi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a). \tag{16}$$

The ground state corresponds to $n = 1$, $\psi_1(x) = \sqrt{\frac{2}{a}}\sin(\pi x/a)$, for $0 < x < a$ and $\psi_1(x) = 0$ otherwise.

ii) Just after the wall moves, the energy of the system is measured. Assuming the wave function does not change in response to the motion of the wall (since it is very fast), what is the probability of finding the system in the ground state (of the “new” hamiltonian).

The eigenfunctions of the new hamiltonian are given as above after the substitution $a \to L$:

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L). \tag{17}$$

In order to find the probabilities of a measurement of the energy, we have to expand the wave function as a linear combination of the eigenfunction of the hamiltonian

$$\psi_1(x) = \sum_{n=1}^{\infty} c_n\phi_n(x) \Rightarrow c_n = \int_0^a \phi_n^*(x)\psi_1(x) = \int_0^a \sqrt{\frac{2}{L}}\sin(n\pi x/L)\sqrt{\frac{2}{a}}\sin(\pi x/a). \tag{18}$$

The probability we are looking for is given by

$$|c_1|^2 = \int_0^a \sqrt{\frac{2}{L}}\sin(\pi x/L)\sqrt{\frac{2}{a}}\sin(\pi x/a) = |\frac{2\sqrt{aL^3}\sin(a\pi/L)}{\pi}\frac{1}{L^2-a^2}|^2 = \frac{4aL^3\sin^2(a\pi/L)}{\pi^2}\left(\frac{1}{L^2-a^2}\right)^2$$

iii) After the measurement of the energy finds teh ground state energy, the wave function collapses to that of the ground state and a subsequent measurement of the energy will return teh ground state energy with 100% probability.

iii) suppose the system was found to be in the ground state (of the “new” hamiltonian) in item ii). If the energy is measured again, what is the probability of finding the system in the ground state again.