2. a) $\Psi = \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\psi_1$

$$\hat{a}_\pm = \frac{m\omega\hat{x} \mp i\hat{p}}{\sqrt{2\hbar m\omega}} \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

$$\langle x\rangle = \int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}}\psi_0^* - \frac{1}{\sqrt{2}}\psi_1^*\right)\sqrt{\frac{\hbar}{2m\omega}}\,\underbrace{(\hat{a}_+ + \hat{a}_-)\left(\frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\psi_1\right)}_{\frac{1}{\sqrt{2}}(\psi_1 - \sqrt{2}\psi_2 - \psi_0)}$$

$$= \frac{1}{2}(-1-1)\sqrt{\frac{\hbar}{2m\omega}} = -\sqrt{\frac{\hbar}{2m\omega}}$$

b) $\langle V\rangle = \frac{m\omega^2}{2}\langle x^2\rangle$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}}\psi_0^* - \frac{1}{\sqrt{2}}\psi_1^*\right)\overbrace{\frac{\hbar}{2m\omega}(\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)}^{\hat{x}^2}\left(\frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\psi_1\right)$$

$$= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}}\psi_0^* - \frac{1}{\sqrt{2}}\psi_1^*\right)\left(\frac{1}{\sqrt{2}}\sqrt{2}\psi_2 - \sqrt{6}\,\psi_3 - \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\sqrt{2}\sqrt{2}\,\psi_1\right)$$

$$= \frac{\hbar}{2m\omega}\,\frac{1}{2}(1+1+2) = \frac{\hbar}{m\omega}$$

$$\langle V\rangle = \frac{m\omega^2}{2}\frac{\hbar}{m\omega} = \frac{\hbar\omega}{2}$$

c) $\varphi_{0,1}$ are stationary states and evolve in time as $\varphi_n(x)\, e^{-iE_n t/\hbar}$.

By the linearity of the Schrödinger eq. we have

$$\Psi(x,t) = \frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar} \varphi_0(x) - \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar} \varphi_1(x)$$

$$= \frac{1}{\sqrt{2}} e^{-i\omega t/2} \varphi_0(x) - \frac{1}{\sqrt{2}} e^{-i3\omega t/2} \varphi_1(x)$$

d) $$\langle \hat{H} \rangle = \int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}} e^{i\omega t/2} \varphi_0^*(x) - \frac{1}{\sqrt{2}} e^{i3\omega t/2} \varphi_1^*(x) \right)$$

$$\underbrace{\hat{H} \left(\frac{1}{\sqrt{2}} e^{-i\omega t/2} \varphi_0(x) - \frac{1}{\sqrt{2}} e^{-i3\omega t/2} \varphi_1(x) \right)}_{\frac{1}{\sqrt{2}} \underbrace{\frac{\hbar\omega}{2}}_{E_0} e^{-i\omega t/2} \varphi_0(x) - \frac{1}{\sqrt{2}} \underbrace{\frac{3\hbar\omega}{2}}_{E_1} e^{-i3\omega t/2} \varphi_1(x)}$$

$$= \frac{1}{2} \frac{\hbar\omega}{2} + \frac{1}{2} \frac{3\hbar\omega}{2} = \hbar\omega \quad \text{(independent of time)}$$

e) Again, $\langle V \rangle = \frac{m\omega^2}{2} \langle x^2 \rangle$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\, \hat{x}^2\, \Psi(x,t) = \int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}} e^{i\omega t/2} \varphi_0^*(x) - \frac{1}{\sqrt{2}} e^{i3\omega t/2} \varphi_1^*(x) \right)$$

$$\frac{\hbar}{2m\omega} \left(\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ \right)$$

$$\left(\frac{1}{\sqrt{2}} e^{-i\omega t/2} \varphi_0(x) - \frac{1}{\sqrt{2}} e^{-i3\omega t/2} \varphi_1(x) \right)$$

from the $(\psi_1\sqrt{2})^2$ in the wave function

$$= \frac{1}{2}\frac{\hbar}{2m\omega}\int_{-\infty}^{\infty} dx \left(e^{\frac{i\omega t}{2}}\psi_0^* + e^{\frac{i3\omega t}{2}}\psi_1^*\right)\Big(e^{-i\omega t/2}\left(\sqrt{2}\,\psi_2 + \psi_0\right)$$

$$- e^{-3i\omega t/2}\left(\sqrt{6}\,\psi_3 + \psi_1 + \sqrt{2}^2\,\psi_1\right)$$

$$= \frac{\hbar}{4m\omega}\left[1 + 3\right] = \frac{\hbar}{m\omega}$$

$$\langle V\rangle = \frac{m\omega^2}{2}\langle x^2\rangle = \frac{m\omega^2}{2}\frac{\hbar}{m\omega} = \frac{\hbar\omega}{2}$$ (independent of Time)

1. a) $1 = \int_{-\infty}^{\infty} dx \; |\psi(x)|^2 = \int_{-\infty}^{\infty} dx \; |A|^2 \, e^{-2\alpha x^2} = |A|^2 \sqrt{\frac{\pi}{2\alpha}}$

$$\Rightarrow A = \left(\frac{2\alpha}{\pi}\right)^{1/4} \Rightarrow \boxed{\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2 - i\beta x}}$$

b) $\langle x \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x) \, x \, \psi(x) = A^2 \int_{-\infty}^{\infty} dx \; e^{-\alpha x^2 + i\beta x} \, x \, e^{-\alpha x^2 - i\beta x}$

$$= A^2 \int_{-\infty}^{\infty} dx \; e^{-2\alpha x^2} x = 0 \quad (\text{odd integrand})$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x) \, x^2 \, \psi(x) = A^2 \int_{-\infty}^{\infty} dx \; e^{-2\alpha x^2} x^2 = \sqrt{\frac{2\alpha}{\pi}} \sqrt{\frac{\pi}{2^5 \alpha^5}} \, \frac{4\alpha}{4}$$

$$= \sqrt{\frac{1}{16\alpha^2}} \, \alpha = \frac{1}{4\alpha}$$

$$\boxed{\sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{2\sqrt{\alpha}}}$$

c) $\text{probability} = \int_{0}^{\infty} dx \; |\psi(x)|^2 = \int_{0}^{\infty} dx \; A^2 \, e^{-2\alpha x^2}$

$$= \frac{\int_0^{\infty} dx \; e^{-2\alpha x^2}}{\int_{-\infty}^{\infty} dx \; e^{-2\alpha x^2}} = \frac{1}{2} \quad (\text{symetric integrand})$$

d) $$\langle p\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, \hat{p}\, \psi(x) = A^2 \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + i\beta x} \left(-i\hbar \frac{d}{dx}\right) e^{-\alpha x^2 - i\beta x}$$

$$= -i\hbar |A|^2 \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + i\beta x} \left(-2\alpha x - i\beta\right) e^{-\alpha x^2 - i\beta x}$$

$$= +i\hbar A^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} \left(\underbrace{2\alpha x}_{0} + i\beta\right)$$

(odd integrand)

$$= -\beta\hbar A^2 \underbrace{\int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2}}_{1/A^2} = -\beta\hbar$$

$$\boxed{\langle p\rangle = -\beta\hbar}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, \hat{p}^2\, \psi(x) = A^2 \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + i\beta x} \left(-\hbar^2 \frac{d^2}{dx^2}\right) e^{-\alpha x^2 - i\beta x}$$

$$= -\hbar^2 A^2 \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + i\beta x} \frac{d}{dx}\left((-2\alpha x - i\beta)\, e^{-\alpha x^2 - i\beta x} \right)$$

$$= -\hbar^2 A^2 \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + i\beta x} \underbrace{\left(-2\alpha + (-2\alpha x - i\beta)^2\right)}_{-2\alpha + 4\alpha^2 x^2 + 4i\alpha\beta x - \beta^2} e^{-\alpha x^2 - i\beta x}$$

$$= -\hbar^2 A^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} \left(4\alpha^2 x^2 + (-2\alpha - \beta^2 + \underbrace{4i\alpha\beta x}_{\text{odd integrand}})\right)$$

$$= -\hbar^2 \sqrt{\frac{2\alpha}{\pi}} \left[\underbrace{4\alpha^2 \sqrt{\frac{\pi}{32\alpha^5}}\, \alpha}_{\alpha\sqrt{\frac{\pi}{2\alpha}}} - (2\alpha + \beta^2) \sqrt{\frac{\pi}{2\alpha}} \right]$$

$$= -\hbar^2 \sqrt{\frac{2\alpha}{\pi}} \left[-\alpha - \beta^2 \right] \sqrt{\frac{\pi}{2\alpha}} = \hbar^2(\alpha + \beta^2) \Rightarrow \boxed{\langle p^2\rangle = \hbar^2(\alpha + \beta^2)}$$

e) $$\underbrace{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}_{\sigma_x}\ \underbrace{\sqrt{\langle p^2\rangle - \langle p\rangle^2}}_{\sigma_p} = \frac{1}{2\sqrt{\alpha}}\sqrt{\hbar^2(\alpha+\beta^2) - \beta^2\hbar^2}$$

$$= \frac{1}{2\sqrt{\alpha}}\,\hbar\sqrt{\alpha}$$

$$= \frac{\hbar}{2} \geq \frac{\hbar}{2}$$

the uncertainty principle is respected