QUANTUM PHYSICS I SUMMARY
(UP TO 1ST MIDTERM)

GENERAL FORMALISM

- state of the particle at a particular time described by a <u>complex</u> function of the position : the wave function $\Psi(x)$.

- every classical observable corresponds to an operator

position: $x \Rightarrow \hat{x}$, $\hat{x}\,\varphi(x) = x\,\varphi(x)$

momentum: $p \Rightarrow \hat{p}$, $\hat{p}\,\varphi(x) = -i\hbar \frac{d}{dx}\varphi(x)$

kinetic energy : $T = \frac{p^2}{2M} \Rightarrow \hat{T} = \frac{\hat{p}^2}{2M}$, $\hat{T}\,\varphi(x) = -\frac{\hbar^2}{2M}\frac{d^2}{dx^2}\varphi(x)$

total energy (hamiltonian) : $H = \frac{p^2}{2M} + V(x) \Rightarrow \hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x})$, $\hat{H}\,\varphi(x) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V(x)\right]\varphi(x)$

⋮

- these operators have a set of eigenvalue/eigenfunction pairs

$$\hat{A}\,\varphi_n(x) = a_n\,\varphi_n(x)$$

eigenvalue

eigenfunction

(sometimes n's form a discrete set, sometimes a continuous set)

Examples:

momentum: $\hat{p}\,\varphi_p(x) = p\,\varphi_p(x) \Rightarrow \varphi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$, p = real number

position: $\hat{x}\,\varphi_{x_0}(x) = x_0\,\varphi_{x_0}(x) \Rightarrow \varphi_{x_0}(x) = \delta(x - x_0)$, x_0 = real number

hamiltonian for free particle: $\underbrace{\hat{H}}_{\hat{p}^2/2M}\varphi_k(x) = E_k\,\varphi_k(x) \Rightarrow \varphi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}}$, $E_k = \frac{\hbar^2 k^2}{2M}$ for any real k

hamiltonian of the infinite square well : $\underbrace{\hat{H}}_{\frac{\hat{p}^2}{2M}+V(x)} \varphi_n(x) = E_n \varphi_n(x) \Rightarrow \varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2 \hbar^2 \pi^2}{2ML^2}$

for $n = 1, 2, \ldots$

$V(x)$: 0 for $0<x<L$, ∞ otherwise

in all these examples the eigenfunctions satisfy an orthonormality condition:

discrete spectrum
$$\int_{-\infty}^{\infty} dx \; \varphi_n^*(x)\, \varphi_m(x) = \delta_{nm} = \begin{cases} 1 & \text{for } n=m \\ 0 & \text{for } n \neq m \end{cases}$$

OR

continuous spectrum
$$\int_{-\infty}^{\infty} dx \; \varphi_p^*(x)\, \varphi_{p'}(x) = \delta(p-p')$$

- The wave function evolves in time according to the (time-dependent) Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \underbrace{-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \Psi(x,t) + V(x)\, \Psi(x,t)}_{\hat{H}\, \Psi(x,t)}$$

The general solution of the Schrödinger eq. is given by

$$\Psi(x,t) = \sum_n c_n \, e^{-iE_n t/\hbar} \, \varphi_n(x),$$

where E_n, $\varphi_n(x)$ are the eigenvalues/eigenfunctions of $\hat{H}$

$$\hat{H}\, \varphi_n(x) = E_n \, \varphi_n(x)$$

The c_n's are determined by the initial condition

$$\Psi(x,0) = \sum_n c_n \varphi_n(x) \Rightarrow c_n = \int_{-\infty}^{\infty} dx \; \varphi_n^*(x)\, \Psi(x,0)$$

- the only outcomes of a measurement of an observable $A(x,p)$ at time t are the eigenvalues of $\hat{A}$. The probability of getting one particular eigenvalue a_n is given by $|c_n|^2$ where c_n is the coefficient of the expansion of Ψ in terms of φ_n's.

$$\hat{A}\,\varphi_n(x) = a_n\,\varphi_n(x)$$

PROBABILITY OF GETTING THIS EIGENVALUE IS THE MAGNITUDE SQUARE OF THIS

$$\Psi(x,t) = \sum_n c_n\,\varphi_n(x) \Rightarrow c_n = \int_{-\infty}^{\infty} dx\,\varphi_n^*(x)\,\Psi(x,t)$$

Examples:

1) measurement of energy on a infinite square well when the wave function is

$$\Psi(x,t) = \underbrace{\sqrt{\frac{3}{5}}}_{c_1}\underbrace{\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}}_{\varphi_1(x)} - \underbrace{i\sqrt{\frac{2}{5}}}_{c_3}\underbrace{\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}}_{\varphi_3(x)}$$

probability for $E_1 = \frac{\hbar^2\pi^2}{2mL^2}$ $= |c_1|^2 = 3/5$

probability for $E_3 = 9\,\frac{\hbar^2\pi^2}{2mL^2}$ $= |c_3|^2 = 2/5$

any other E_n $= |c_n|^2 = 0 \quad (n \neq 1,3)$

2) measurement of momentum when the wave function is $\Psi(x,t) = \varphi(x)$

$$\varphi(x) = \int_{-\infty}^{\infty} dp \, \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{2\pi\hbar}} \, C(p) \Rightarrow C(p) = \int_{-\infty}^{\infty} dx \, \frac{e^{-i\frac{p}{\hbar}x}}{\sqrt{2\pi\hbar}} \, \varphi(x) = \text{"Fourier Transform of } \varphi(x)\text{"}$$

$\varphi(x)$ written as a linear combination of momentum eigenfunctions

probability (density) of measuring p is $= |C(p)|^2 = \left| \int_{-\infty}^{\infty} dx \, \frac{e^{-i\frac{p}{\hbar}x}}{\sqrt{2\pi\hbar}} \, \varphi(x) \right|^2$

3) measurement of position when the wave function is $\Psi(x,t) = \varphi(x)$

$$\varphi(x) = \int_{-\infty}^{\infty} dy \; C(y) \, \delta(x-y) \Rightarrow C(y) = \int_{-\infty}^{\infty} dx \; \delta(x-y) \, \varphi(x) = \varphi(y)$$

$\varphi(x)$ written as a linear combination of position eigenfunctions

probability (density) of finding the particle at y $= |C(y)|^2 = |\varphi(y)|^2$

It follows from the rule above that the <u>average</u> value of the measurement of A is given by

$$\langle A \rangle = \int_{-\infty}^{\infty} dx \; \Psi^*(x,t) \, \hat{A} \, \Psi(x,t) \, .$$