# PHY 401 - QUANTUM PHYSICS

## HW 8 - SOLUTION

A. i) $\left[ -\frac{\hbar^2}{2M} \nabla^2 + V(r,\phi,z) \right] \psi(r,\phi,z) = E\, \psi(r,\phi,z)$

$$-\frac{\hbar^2}{2M} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \psi + V\psi = E\,\psi$$

separable ansatz: $\psi(r,\phi,z) = R(r)\, \Phi(\phi)\, Z(z)$

$$-\frac{\hbar^2}{2M} \left( \frac{1}{R}\frac{d^2}{dr^2}R + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2}\underbrace{\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi}_{-m^2 = \text{constant}} + \underbrace{\frac{1}{Z}\frac{d^2}{dz^2}Z}_{-K^2 = \text{constant}} \right) + V = E$$

$\Downarrow$

$$Z''(z) = -K^2 Z(z)$$

$$\Phi''(\phi) = -m^2 \Phi(\phi)$$

$$R''(r) + \frac{1}{r}R'(r) - \frac{m^2}{r^2}R - K^2 R - \frac{2MVR}{\hbar^2} = -\frac{2MER}{\hbar^2}$$

ii) $Z'' + k^2 Z = 0, \quad Z(0) = Z(L) = 0$

↳ wave function vanishes at the bottom and at the top of the cylinder

⇩

$Z(z) = \sin kz$, (not normalized)

$k = \frac{2\pi n}{L}, \quad n = 1, 2, \ldots$

$\Phi'' + m^2 \Phi = 0, \quad \Phi(0) = \Phi(2\pi)$ (wave function at $\phi = 0$ and $\phi = 2\pi$ is the same)

⇩

$\Phi(\phi) = e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \ldots$ (not normalized)

iii) Inside the cylinder, $V = 0$, so

$$R'' + \frac{1}{r} R' + \left( \frac{2ME}{\hbar^2} - k^2 - \frac{m^2}{r^2} \right) R = 0$$

or

$$r^2 R'' + r R' + \left( r^2 (k_0^2 - k^2) - m^2 \right) R = 0, \quad \text{with } k_0^2 = \frac{2ME}{\hbar^2}.$$

or

① $$x^2 \frac{d^2 R(x)}{dx^2} + x \frac{dR(x)}{dx} + (x^2 - m^2) R(x) = 0, \quad \text{with } x = r\sqrt{k_0^2 - k^2}$$

Eq. ① is (one of) the standard form of the Bessel equation. Its solution is

$$R(x) = A J_m(x) + B Y_m(x).$$

At small $x$, $Y_m(x) \xrightarrow[x \to 0]{} \begin{cases} \frac{2}{\pi} \ln x/2, & \text{for } m = 0 \\ -\frac{\Gamma(m)}{\pi} \left( \frac{2}{x} \right)^m, & \text{for } m \neq 0 \end{cases} \to -\infty$

$Y_m(x)$ is not acceptable because ~~[crossed out]~~ $\psi(r \to 0)$ should be finite

so $B = 0$.

iv) boundary condition at $r=\rho$:

$$R(r=\rho) = A\ J_m\left(\sqrt{k_0^2-k^2}\,\rho\right) = 0 \Rightarrow (k_0^2-k^2)\rho^2 = \beta_{mn}$$

↑ zeroes of the $m^{th}$ Bessel function

For s-waves $(m=0)$

$$(k_0^2-k^2)\rho^2 = \beta_{0n} = \begin{cases} 2.40\ldots & , n=1 \\ 5.52\ldots & , n=2 \\ 8.65\ldots & , n=3 \\ 11.79\ldots & , n=4 \end{cases}$$

⇓

$$\frac{2ME}{\hbar^2} = \frac{\beta_{0n}}{\rho^2} + k^2 = \frac{\beta_{0n}}{\rho^2} + \left(\frac{2\pi}{L}\right)^2 n'^2$$

this is a different "n" from this

$$E_{nn'} = \frac{\hbar^2}{2M}\left(\frac{\beta_{0n}}{\rho^2} + \frac{4\pi^2}{L^2}n'^2\right),$$ with $\beta_{0n}$ as above and $n' = 1, 2, \ldots$

B. i) $$Y_{\ell m}(\theta,\phi) = A\ e^{im\phi}\ P_\ell^m(\cos\theta)$$

↑ some normalization constant

$$\hat{L}_z Y_{\ell m}(\theta,\phi) = A\left(-i\hbar\frac{d}{d\phi}\right)e^{im\phi}\ P_\ell^m(\cos\theta)$$

$$= \hbar m\ A e^{im\phi}\ P_\ell^m(\cos\theta)$$

$$= \underbrace{\hbar m}_{\text{eigenvalue}}\ Y_{\ell m}(\theta,\phi).$$

ii) $Y_{1-1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\, e^{-i\phi} \sin\theta$

$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}\right)$

$$\hat{L}^2\, Y_{1-1}(\theta,\phi) = (-\hbar^2)\,-\sqrt{\frac{3}{8\pi}}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}\right] e^{-i\phi}\sin\theta$$

$$= (-\hbar^2)\,-\sqrt{\frac{3}{8\pi}}\left[e^{-i\phi}\frac{1}{\sin\theta}\underbrace{\frac{d}{d\theta}(\sin\theta\cos\theta)}_{\underbrace{\cos^2\theta-\sin^2\theta}_{1-2\sin^2\theta}} + \frac{1}{\sin\theta}(-i)^2 e^{-i\phi}\right]$$

$$= (-\hbar^2)\,-\sqrt{\frac{3}{8\pi}}\, e^{-i\phi}\left[\frac{1}{\sin\theta} - 2\sin\theta - \frac{1}{\sin\theta}\right]$$

$$= \underbrace{2\hbar^2}_{\text{eigenvalue}}\ \underbrace{\left[-\sqrt{\frac{3}{8\pi}}\, e^{-i\phi}\sin\theta\right]}_{Y_{1-1}(\theta,\phi)}$$

eigenvalue $= 2\hbar^2 = \hbar^2 1(1+1)$

iii) $e^{-i\frac{\varphi}{\hbar}\hat{L}_z}\,\psi(r,\theta,\phi) = e^{-\varphi\frac{d}{d\phi}}\,\psi(r,\theta,\phi)$

$$= \left[1 - \varphi\frac{d}{d\phi} + \frac{1}{2!}(-\varphi)^2\frac{d^2}{d\phi^2} + \cdots\right]\psi(r,\theta,\phi)$$

$$= \underbrace{\psi(r,\theta,\phi) - \varphi\frac{\partial}{\partial\phi}\psi(r,\theta,\phi) + \frac{1}{2!}(-\varphi)^2\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi)}_{\text{Taylor series}} + \cdots$$

$$= \psi(r,\theta,\phi-\varphi)$$

↖ Sorry for the sign mistake!