QUANTUM PHYSICS I

PROBLEM SET 6

SOLUTION

A. Eigenfunctions and eigenvalues of common operators

What are the eigenfunction and eigenvalues of the operators

i) $\hat{x}$

As seen in class, the eigenfunctions are $f_{x_0}(x) = \delta(x - x_0)$ with eigenvalues x_0, for any real x_0. In fact,

$$\hat{x} f_{x_0}(x) = \hat{x}\delta(x - x_0) = x\delta(x - x_0) = x_0\delta(x - x_0) = x_0 f_{x_0}(x). \tag{1}$$

ii) $\hat{p}$ As seen in class, the eigenfunctions are $f_k(x) = e^{ikx}/\sqrt{2\pi}$ with eigenvalues $\hbar k$, for any real k. In fact,

$$\hat{p} f_k(x) = -i\hbar \frac{d}{dx}\frac{e^{ikx}}{\sqrt{2\pi}} = \hbar k \frac{e^{ikx}}{\sqrt{2\pi}} = \hbar k f_k(x). \tag{2}$$

Repeat items i) and ii).

B. Eigenfunctions of kinetic energy

What are the eigenfunctions and eigenvalues of the kinetic operator $\hat{K} = \hat{p}^2/2m$. Show two degenerate eigenfunctions of the kinetic operator which are orthogonal to each other. Also, show two degenerate eigenfunctions that are NOT orthogonal.

The eigenfunctions of $\hat{K}$ are the same as the ones of $\hat{p}$:

$$\hat{K} f_k(x) = \frac{1}{2m}\hat{p}\hat{p} f_k(x) = \frac{1}{2m}\hat{p}\hbar k f_k(x) = \frac{1}{2m}(\hbar k)^2 f_k(x), \tag{3}$$

and the corresponding eigenvalues are $\hbar^2 k^2/2m$. The functions $f_k(x)$ and $f_{-k}(x)$ are two linearly independent eigenfunctions (as long as $k \neq 0$) since they share the same eigenvalue $\hbar^2 k^2/2m$. They are also orthogonal since

$$\int_{-\infty}^{\infty} dx\ (f_{-k}(x))^* f_k(x) = \int_{-\infty}^{\infty} dx\ \frac{1}{2\pi} e^{ikx} e^{ikx} = \delta(2k) = 0, \tag{4}$$

where we assumed that $k \neq 0$. An example of two non-orthogonal degenerate eigenfunctions is $f_k(x)$ and $17 f_k(x) - 3i f_{-k}(x)/\sqrt{17^2 + 3^2}$.

C. Schroedinger equation in momentum space

Denote by $|k\rangle$ the momentum eigenfunction with eigenvalue $p = \hbar k$, that is

$$\hat{p}|k\rangle = \hbar k|k\rangle, \tag{5}$$

and by $|n\rangle$ the energy eigenfunction of the hamiltonian $\hat{H} = \hat{p}^2/2m + \hat{V}$ with eigenvalue E_n

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{6}$$

Write the time independent Schroedinger equation in 6 in the basis of momentum eigenfunctions. You should obtain an equation for $\psi_n(k) = \langle k|n\rangle$ depending on the "matrix element" $\langle k|\hat{V}|k'\rangle$. Hint: the answer is

$$\psi_n(k)\left[\frac{\hbar^2 k^2}{2m} - E_n\right] + \int dk' \ \langle k|\hat{V}|k'\rangle\psi_n(k') = 0. \tag{7}$$

Multiplying the Schr´odinger on the left by $\langle k|$ and inserting the identity $\mathbb{1} = \int dk'|k'\rangle\langle k'|$ we have

$$\begin{aligned}
\langle k|\hat{H}\int dk'|k'\rangle\langle k'|n\rangle &= \langle k|E_n|n\rangle \\
\int dk' \ \langle k|\frac{\hat{p}^2}{2m} + V(\hat{x})|k'\rangle\langle k'|n\rangle &= E_n\langle k|n\rangle \\
\int dk' \ \left[\frac{k^2}{2m}\underbrace{\langle k|k'\rangle}_{\delta(k-k')} + \langle k|V(\hat{x})|k'\rangle\right]\langle k'|n\rangle &= E_n\langle k|n\rangle.
\end{aligned} \tag{8}$$

Defining $\psi_n(k) = \langle k|n\rangle$ we then have

$$\psi_n(k)\left[\frac{\hbar^2 k^2}{2m} - E_n\right] + \int dk' \ \langle k|\hat{V}|k'\rangle\psi_n(k') = 0. \tag{9}$$

D. Probabilistic interpretation

A particle lives inside two impenetrable walls at $x = 0$ and $x = a$. Its wave function at time t is given by

$$\Psi(x,t) = \begin{cases} \frac{\sqrt{2}}{a}x, \text{for } 0 < x < a, \\ 0, \text{otherwise} \end{cases}. \tag{10}$$

i) If the energy is measured, what are the possible outcomes and with which probabilities?

First, let us normalize the wave function properly:

$$1 = \int_0^a A^2 x^2 = \frac{a^3 A^2}{3} \to A = \sqrt{\frac{3}{a^3}}. \tag{11}$$

The possible outcomes of an energy easuremnt are the eigenvalues of the infinite square well hamiltonian, namely

$$E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2, n = 1, 2, \cdots . \tag{12}$$

To find the probabilities of each one we write the wave function as a linear superposition of eigenfucntions of the hamiltonian

$$Ax = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin(n\pi x/a). \tag{13}$$

The values of the constants c_n are given by

$$c_n = \int_0^a \sqrt{\frac{3}{a^3}} x \sqrt{\frac{2}{a}} \sin(n\pi x/a) = -\sqrt{6}\frac{(-1)^n}{n\pi}. \tag{14}$$

The probability of finding the value E_n is given by $|c_n|^2 = 6/(n^2\pi^2)$.

ii) Suppose the energy measurement results in the value $E = 4\hbar^2\pi^2/(2ma^2)$. What is the expected (average) value of the position *immediately after* this measurement?

If the energy value E_2 is measured, the wave function collapses to $\psi_2(x) = \sqrt{\frac{2}{a}} \sin(2\pi x/a)$. A subsequent measurement of the position will yield the value x with probability $|\sqrt{\frac{2}{a}} \sin(2\pi x/a)|^2$. The expected value will be

$$\int_0^a dx\; x \frac{2}{a} \sin^2(2\pi x/a) = \frac{a}{2}. \tag{15}$$