QUANTUM PHYSICS I

PROBLEM SET 5 - SOLUTION

due November 10^{th} before class

A. Muonic hydrogen

A muon is a particle identical to an electron except its mass is about 200 times larger. A muonic hydrogen is a bound state of a proton to a muon (instead of a proton and an electron as in the usual hydrogen).

i) Use Bohr's theory to calculate the energy levels of the muonic hydrogen. What is the energy of the ground state (in eV) ?

We proceed as we (well, Bohr) did in the regular hydrogen atom. We use Newton's law

$$\underbrace{\frac{e^2}{4\pi\epsilon_0 r^2}}_{F} = \underbrace{m_\mu \frac{v^2}{r}}_{ma} \tag{1}$$

and the Bohr quantization condition

$$L = m_\mu v r = n\hbar, \tag{2}$$

where $n = 1, 2 \cdots$. Eliminating v in the first equation and plugging in the second we can find r

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{m_\mu e^2} n^2. \tag{3}$$

The energy in each orbit can be found by

$$E_n = \frac{m_\mu v^2}{2} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{m_\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2} = \frac{m_\mu}{m_e} E_n^{hydrogen} \approx 200 \times (-13.6\ eV) \approx -2.72 keV, \tag{4}$$

where in the last step we restrict ourselves to the $n = 1$ case. Note that, in the calculation above, we assume that the nucleus does not move. Due to the difference in the masses of the proton and electron ($m_p \approx 2000 m_e$) that was an excellent approximation for the hydrogen atom. here, the difference between the proton and muon mass is not so large so this approximation is not so good.

ii) Is the muon non-relativistic in the ground state of the muonic hydrogen ?

The rest mass of the muon is about $200 \times m_e \approx 200 \times 500\ keV$, which is much larger than the binding (or kinetic) energy of the muon in the muonic hydrogen. The muon is as non-relativistic here and the electron is in the hydrogen atom (there are, however relativistic corrections to the hydrogen spectrum and they are easily measured).

iii) What is the wavelength of a photon emitted in a transition between the first excited state and the ground state ? Which kind of photon is it (radio, microwave, visible, ultraviolet, X-ray, γ-ray, ...) ?

The energy of the photon equals the difference in energy between the ground and first excited state

$$E_\gamma = 2.72 keV \left(\frac{1}{1^2} - \frac{1}{2^2}\right) \approx 2.04 keV. \tag{5}$$

Using $E = h\nu$ and $\lambda = c/\nu$ we find the wavelength to be

$$\lambda = \frac{hc}{E} \approx 0.6\ nm, \tag{6}$$

which is in the X-ray region.

iv) Besides the mass, muons differ from electron by the fact that they "decay" into an electron and two massless, chargeless particles called "neutrinos". The lifetime for the decay is about $2 \times 10^{-6} s$. Does the muon have the time to orbit the proton several times before decaying when it is in the ground state of muonic hydrogen ?

The time it takes for the muon to go around one orbit is $T = r/v$. From 3 we find that $v = n\hbar/(m_\mu r)$ so, for the ground state we have

$$T = \frac{r}{v} = \frac{m_\mu r^2}{\hbar} = \underbrace{\left(\frac{4\pi\epsilon_0 \hbar c}{e^2}\right)^2}_{1/\alpha^2} \frac{1}{\hbar^2 c^2} \frac{\hbar^3}{m_\mu} \approx (137)^2 \frac{4.13/2\pi \times 10^{-15} eV.s}{1 \times 10^6 \frac{eV}{c^2} c^2} \approx 6 \times 10^{-15} s >> \text{muon lifetime}. \tag{7}$$

B. Operator wizardry

i) Show that the operator $\hat{p} = -i\hbar d/dx$ is an hermitian operator.

$$\langle f|\hat{p}|g\rangle = \int_{-\infty}^{\infty} dx\ f^*(x)(-i\hbar\frac{d}{dx})g(x) = \int_{-\infty}^{\infty} dx\ (i\hbar\frac{d}{dx}f^*(x))g(x) = \int_{-\infty}^{\infty} dx\ (-i\hbar\frac{d}{dx}f(x))^* g(x) = \langle \hat{p}f|g\rangle. \quad (8)$$

ii) Compute

$$e^{-i\frac{y}{\hbar}\hat{p}}f(x). \quad (9)$$

Hint: expand the exponential and remember the Taylor series expression.

$$\begin{aligned} e^{-i\frac{y}{\hbar}\hat{p}}f(x) &= (1 - i\frac{y}{\hbar}\hat{p} + \frac{1}{2!}(-i\frac{y}{\hbar}\hat{p})^2 + \cdots)f(x) && (10)\\ &= (1 - y\frac{d}{dx} + \frac{1}{2!}(-y\frac{d}{dx})^2 + \cdots)f(x) && (11)\\ &= f(x) - yf'(x) + \frac{1}{2!}y^2 f''(x) + \cdots && (12)\\ &= f(x-y). && (13) \end{aligned}$$

iii) Show that $\Psi(x,t) = e^{-i\hat{H}t/\hbar}\Psi(x,0)$ satisfies the Schrödinger equation. It is said that $\hat{p}$ generates space translations (from item ii) and $\hat{H}$ generates time translations (by item iii).

$$\begin{aligned} i\hbar\frac{d}{dt}e^{-i\hat{H}t/\hbar}\Psi(x,0) &= i\hbar\frac{d}{dt}\sum_{n=0}^{\infty}\frac{(\frac{-i\hat{H}t}{\hbar})^n}{n!}\Psi(x,0) && (14)\\ &= \sum_{n=1}^{\infty} n\hat{H}\frac{(\frac{-i\hat{H}t}{\hbar})^{n-1}}{n!}\Psi(x,0) && (15)\\ &= \hat{H}\sum_{n'=0}^{\infty}\frac{(\frac{-i\hat{H}t}{\hbar})^{n'}}{n'!}\Psi(x,0) && (16)\\ &= \hat{H}e^{-i\hat{H}t/\hbar}\Psi(x,0). && (17) \end{aligned}$$