Exercise A

Case $E < V_0$**:** First we calculate the wave function $\psi(x)$ from time-independent Schrödinger Equation:

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) & x<0 \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = -(V_0-E)\psi(x) & x>0 \end{cases} \Rightarrow \psi(x) = \begin{cases} Ae^{ikx}+Be^{-ikx} & x<0 \\ Ce^{\kappa x}+De^{-\kappa x} & x>0 \end{cases} \text{ where } k=\sqrt{\frac{2mE}{\hbar^2}},\ \kappa=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

To find coefficients A, B, C, D:

(i) Since $e^{\kappa x} \to \infty$ as $x \to \infty$, $e^{\kappa x}$ doesn't belong to physical Hilbert space, that is, $C = 0$.

(ii) We need $\psi(x)$ to be continuous at $x = 0$, that is, $Ae^0 + Be^0 = De^0 \Rightarrow A + B = D$

(iii) We need $\psi'(x)$ to be continuous at $x = 0$, that is, $ikAe^0 - ikBe^0 = -\kappa De^0 \Rightarrow A - B = \frac{i\kappa}{k}D$

Solve the equations (ii) and (iii), we have $A = \frac{1}{2}\left(1 + \frac{i\kappa}{k}\right)D, B = \frac{1}{2}\left(1 - \frac{i\kappa}{k}\right)D$.
The transmission coefficient is

$$T = \left|\frac{j_T}{j_I}\right| = 0$$

since the trasmission part of $\psi(x)$ is $\psi_T(x) = De^{-\kappa x}$, which is a real function multiplied by a constant, thus

$$j_T = \frac{\hbar}{2mi}\left(\psi_T^*(x)\psi_T'(x) - \psi_T(x)\psi_T'^*(x)\right) = \frac{\hbar}{2mi}|D|^2\left(-\kappa e^{-2\kappa x} + \kappa e^{-2\kappa x}\right) = 0$$

The reflection coefficient is

$$R = \frac{|B|^2}{|A|^2} = \frac{\left|1 - \frac{i\kappa}{k}\right|^2}{\left|1 + \frac{i\kappa}{k}\right|^2} = \frac{k^2+\kappa^2}{k^2+\kappa^2} = 1$$

Thus, $R + T = 1$, verified.

Case $E > V_0$**:** For the case $E > V_0$, we just need to adjust the $x > 0$ part a little bit:

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) & x<0 \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = (E-V_0)\psi(x) & x>0 \end{cases} \Rightarrow \psi(x) = \begin{cases} Ae^{ikx}+Be^{-ikx} & x<0 \\ Ce^{ik'x}+De^{-ik'x} & x>0 \end{cases} \text{ where } k=\sqrt{\frac{2mE}{\hbar^2}},\ k'=\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

To find coefficients A, B, C, D:

(i) Since e^{-ikx} implies a backward-going wave function, and there is no such a part for $x > 0$, that is, $D = 0$.

(ii) We need $\psi(x)$ to be continuous at $x = 0$, that is, $Ae^0 + Be^0 = Ce^0 \Rightarrow A + B = C$

(iii) We need $\psi'(x)$ to be continuous at $x = 0$, that is, $ikAe^0 - ikBe^0 = ik'Ce^0 \Rightarrow A - B = \frac{k'}{k}C$

Solve the equations (ii) and (iii), we have $A = \frac{1}{2}\left(1 + \frac{k'}{k}\right)C, B = \frac{1}{2}\left(1 - \frac{k'}{k}\right)C$.
The transmission coefficient is

$$T = \frac{k'|C|^2}{k|A|^2} = \frac{\frac{k'}{k}}{\left|\frac{1}{2}\left(1+\frac{k'}{k}\right)\right|^2} = \frac{4\sqrt{E(E-V_0)}}{|\sqrt{E}+\sqrt{E-V_0}|^2}$$

The reflection coefficient is

$$R = \frac{|B|^2}{|A|^2} = \frac{\left|1-\frac{k'}{k}\right|^2}{\left|1+\frac{k'}{k}\right|^2} = \frac{|\sqrt{E}-\sqrt{E-V_0}|^2}{|\sqrt{E}+\sqrt{E-V_0}|^2}$$

Thus,

$$R + T = \frac{4\sqrt{E(E-V_0)} + |\sqrt{E}-\sqrt{E-V_0}|^2}{|\sqrt{E}+\sqrt{E-V_0}|^2} = \frac{E + 2\sqrt{E(E-V_0)} + E - V_0}{|\sqrt{E}+\sqrt{E-V_0}|^2} = \frac{|\sqrt{E}+\sqrt{E-V_0}|^2}{|\sqrt{E}+\sqrt{E-V_0}|^2} = 1$$

Verified.

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Exercise B

a) Let $f(x) = (x^3 - 1), y = 1$, then

$$\int_{-\infty}^{\infty} dx(x^3 - 1)\delta(x-1) = \int_{-\infty}^{\infty} dx f(x)\delta(x-y) = f(y) = 1^3 - 1 = 0$$

b) Let $u = cx$, and use integration by parts:

$$\int_{-\infty}^{\infty} dx f(x)\delta(cx) = \begin{cases} \int_{-\infty}^{\infty} \frac{du}{c} f\left(\frac{u}{c}\right)\delta(u) & c \geq 0 \\ \int_{\infty}^{-\infty} \frac{du}{c} f\left(\frac{u}{c}\right)\delta(u) & c < 0 \end{cases} = \int_{-\infty}^{\infty} \frac{du}{|c|} f\left(\frac{u}{c}\right)\delta(u) = \frac{f(0)}{|c|}$$

And

$$\int_{-\infty}^{\infty} dx f(x)\frac{\delta(x)}{|c|} = \frac{f(0)}{|c|}$$

Thus, $\delta(cx) = \frac{1}{|c|}\delta(x)$.

c) Define

$$f(x) = \int_{-\infty}^{x} dt\delta(t)$$

then it can be easily found that

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

which implies $f(x) = \theta(x)$, that is, $\theta(x) = \int_{-\infty}^{x} dt\delta(t)$. Take derivative with respect to x on both sides and apply fundamental theorem of calculus, we will get $\frac{d\theta(x)}{dx} = \delta(x)$.

d)

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\delta(x) = \frac{e^0}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$$

By Plancherel's theorem, we have

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikx} F(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$$

e) Use integration by parts:

$$\int_{-\infty}^{\infty} dx f(x)\delta'(x) = f(x)\delta(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx f'(x)\delta(x) = 0 - f'(0) = -f'(0)$$

f) Use the integral formula $\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \int_{-\infty}^{\infty} dx e^{-a(x-\frac{b}{2a})^2+\frac{b^2}{4a}} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\left(\lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}\right) = \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\alpha x^2 - ikx} = \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{\alpha}} \frac{e^{\frac{(ik)^2}{4\alpha}}}{\sqrt{2\pi}} = \lim_{\alpha\to\infty} \frac{e^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$$

It is the same as the result in d).

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