PHY 401 – QUANTUM PHYSICS – 2008

HW 2 – SOLUTIONS

A. i) $\Psi(x,0) = A(\varphi_1(x) + \varphi_2(x))$

$$1 = \int_0^a dx\, \Psi^*(x,0)\,\Psi(x,0) = \int_0^a dx\, |A|^2 \left(\varphi_1^*(x) + \varphi_2^*(x)\right)\left(\varphi_1(x) + \varphi_2(x)\right)$$

$$= |A|^2 \int_0^a dx \left(|\varphi_1(x)|^2 + |\varphi_2(x)|^2 + \varphi_1^*(x)\,\varphi_2(x) + \varphi_2^*(x)\,\varphi_1(x) \right)$$

orthonormality conditions →

$$= |A|^2 (1 + 1 + 0 + 0) \Rightarrow A = \frac{1}{\sqrt{2}}$$

ii) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(e^{-iE_1 t/\hbar}\,\varphi_1(x) + e^{-iE_2 t/\hbar}\,\varphi_2(x)\right)$

with $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$ and $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$.

$$|\Psi(x,t)|^2 = \frac{1}{2}\left(e^{iE_1 t/\hbar}\varphi_1^*(x) + e^{iE_2 t/\hbar}\varphi_2^*(x)\right)\left(e^{-iE_1 t/\hbar}\varphi_1(x) + e^{-iE_2 t/\hbar}\varphi_2(x)\right)$$

$$= \frac{1}{2}\left(|\varphi_1(x)|^2 + |\varphi_2(x)|^2 + \varphi_1(x)\,\varphi_2(x)\, 2\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\right)$$

$$= \frac{1}{2}\left[\frac{2}{a}\sin^2\frac{\pi x}{a} + \frac{2}{a}\sin^2\frac{2\pi x}{a} + \frac{2}{a}\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\, 2\cos\Big(\underbrace{\frac{3\pi^2\hbar t}{2ma^2}}_{3\omega t}\Big)\right]$$

vi) $\langle x \rangle = \int_0^a dx \left(e^{iE_1t/\hbar}\psi_1(x) + e^{iE_2t/\hbar - i\phi}\psi_2(x) \right) x \left(e^{-iE_1t/\hbar}\psi_1(x) + e^{-iE_2t/\hbar + i\phi}\psi_2(x) \right)$

$$= \int_0^a dx \left[x\,\psi_1^2(x) + x\,\psi_2^2(x) + x\,\psi_1(x)\psi_2(x)\, 2\cos\left(\frac{(E_1-E_2)t}{\hbar} + \phi \right) \right]$$

$$= \frac{a}{2} - \frac{32}{9\pi^2} a \cos(3\omega t + \phi).$$

(same as before)

The effect of the phase $e^{i\phi}$ is to shif $t \to t + \frac{\phi}{3\omega}$.

B. i) $[\hat{x},\hat{p}]f(x) = x\left(-i\hbar\frac{d}{dx}\right)f(x) - \left(-i\hbar\frac{d}{dx}\right)x f(x)$ ← arbitrary function

$$= -i\hbar\, x f'(x) + i\hbar\,(f(x) + x f'(x))$$

$$= i\hbar f(x)$$

$$\Downarrow$$

$$[\hat{x},\hat{p}] = i\hbar$$

ii) $[\hat{x}^2,\hat{p}]f(x) = x^2\left(-i\hbar\frac{d}{dx}\right)f(x) - \left(-i\hbar\frac{d}{dx}\right)x^2 f(x)$ ← arbitrary function

$$= -i\hbar\, x^2 f'(x) + i\hbar\,(2x f(x) + x^2 f'(x))$$

$$= 2i\hbar f(x)$$

$$\Downarrow$$

$$[\hat{x}^2,\hat{p}] = 2i\hbar$$

iii) $[g(\hat{x}),\hat{p}]f(x) = g(x)\left(-i\hbar\frac{d}{dx}\right)f(x) - \left(-i\hbar\frac{d}{dx}\right)g(x)f(x)$

iii) $\langle x \rangle = \int_0^a dx\, A^2 \left(\psi_1(x) e^{+iE_1 t/\hbar} + \psi_2(x) e^{+iE_2 t/\hbar} \right) x \left(\psi_1(x) e^{-iE_1 t/\hbar} + \psi_2(x) e^{-iE_2 t/\hbar} \right.$

$$= \int_0^a dx\, \frac{1}{2} \left(x\psi_1^2(x) + x\psi_2^2(x) + x\psi_1(x)\psi_2(x)\, 2\cos\left((E_1 - E_2)\frac{t}{\hbar} \right) \right)$$

$$= \frac{1}{2} \left(\frac{2}{a}\frac{a^2}{4} + \frac{2}{a}\frac{a^2}{4} + \frac{2}{a}\, 2 \left(-\frac{8a^2}{9\pi^2} \right) \cos\left((E_1 - E_2)\frac{t}{\hbar} \right) \right)$$

$$= \frac{a}{2} - \frac{32}{9\pi^2} a \cos(3\omega t)$$

iv) First, normalize Ψ:

$$1 = \int_0^a dx\, |A|^2 \left(\psi_1(x) + e^{-i\phi}\psi_2(x) \right) \left(\psi_1(x) + e^{i\phi}\psi_2(x) \right)$$

$$= |A|^2 \int_0^a dx \left(\psi_1^2(x) + \psi_2^2(x) + \psi_1(x)\psi_2(x)\, 2\cos\phi \right)$$

$$= |A|^2 (1 + 1 + 0) \Rightarrow A = 1/\sqrt{2}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \left(e^{-iE_1 t/\hbar} \sin\frac{\pi x}{a} + e^{-iE_2 t/\hbar + i\phi} \sin\frac{2\pi x}{a} \right),$$

with $E_n = \pi^2 \hbar^2 n^2 / 2ma^2$.

v) $|\Psi(x,t)|^2 = \frac{1}{2} \left(e^{iE_1 t/\hbar}\psi_1(x) + e^{iE_2 t/\hbar - i\phi}\psi_2(x) \right) \left(e^{-iE_1 t/\hbar}\psi_1(x) + e^{-\frac{iE_2 t}{\hbar} + i\phi}\psi_2(x) \right.$

$$= \frac{1}{2} \left(\psi_1^2(x) + \psi_2^2(x) + \psi_1(x)\psi_2(x)\, 2\cos\left((E_1 - E_2)\frac{t}{\hbar} + \phi \right) \right)$$