QUANTUM PHYSICS I

PROBLEM SET 2

due September 29, 2008

A. (Griffiths, 2.5 and 2.6, sort of ...) When is the wave function phase relevant ?

A particle on an infinite square well has an initial wave function that is an equal superposition of the two first states:

$$\Psi(x,0) = A(\psi_1(x) + \psi_2(x)), \tag{1}$$

where $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$.

i) Normalize $\Psi(x,0)$.

ii) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. To simplify the result, let $\omega = \pi^2\hbar/(2ma^2)$.

iii) Compute $\langle x\rangle$ and notice it is oscillatory. What is the amplitude and angular frequency of this oscillation ?

Although the overall phase of the wave function is of no physical significance (it cancels out whenever you calculate a measurable quantity), the relative phase of the coefficients in eq. (1) does matter. For example, suppose we take

$$\Psi(x,0) = A(\psi_1(x) + e^{i\phi}\psi_2(x)) \tag{2}$$

instead of eq. (1). Find

iv) $\Psi(x,t)$,

v) $|\Psi(x,t)|^2$ and

vi) $\langle x\rangle$

and compare with the $\phi = 0$ case.

B. Playing with operators

The *commutator* $[\hat{A},\hat{B}]$ of two operators $\hat{A}$ and $\hat{B}$ is defined as the operator $\hat{A}\hat{B} - \hat{B}\hat{A}$. That is, $[\hat{A},\hat{B}]$ acts on a function $\psi(x)$ as $\hat{A}\hat{B}\psi(x) - \hat{B}\hat{A}\psi(x)$. Show that

i) $[\hat{x},\hat{p}] = i\hbar$,

ii) $[\hat{x}^2,\hat{p}] = 2i\hbar\hat{x}$,

iii)$[g(\hat{x}),\hat{p}] = i\hbar\frac{dg(\hat{x})}{dx}$,

where $\hat{p} = -i\hbar d/dx$, $\hat{x} = x$ and $g(x)$ is a well behaved function.

C. Stationary states by numerical methods

In this problem we will find the low lying bound (negative energy) stationary states of a particle moving in the potential

$$V(x) = 10\left(1 - \frac{2}{1+\tanh^2(\lambda x)}\right). \tag{3}$$

This problem, like most, is not analytically solvable.

In order to find stationary states we have to solve the time-independent Schroedinger equation

$$-\frac{d^2\psi_n(x)}{dx^2} + \frac{2m}{\hbar^2}V(x)\psi(x) = \frac{2m}{\hbar^2}E_n\psi_n(x), \tag{4}$$

with the boundary conditions $\psi(-\infty) = \psi(\infty) = 0$. E_n and $\psi_n(x)$ are the unknowns that we want to find out. For generic values of E_n there is no $\psi(x)$ satisfying those conditions.

One way of doing this is to guess a value for E_n and solve the equation with the condition $\psi(-\infty) = 0$. In general, this solution will not satisfy $\psi(\infty) = 0$. We can then change E_n until we find a magic value for which both $\psi(-\infty) = 0$ and $\psi(\infty) = 0$ are fulfilled. You will notice that, for some values of E_n, $\psi(\infty)$ goes to $+\infty$, for some it goes to $-\infty$.

That shows that the value of E_n you are after is in between those two. You can then try to bracket the values of E_n that leads to $\psi(\infty) = 0$. For this reason this method is sometimes called the "wag the tail" method. In the class website you can find a little Mathematica code solving the Schroedinger equation with $\psi(-\infty) = 0$ both for even and odd solutions. Use that to find the energy levels and wavefunctions of the three lowest energy stationary states. Take $\lambda = 1$.

i) In order to have a reasonable guess for the initial value of the energy to try, it's useful to estimate the energy levels first. Use the uncertainty principle to estimate the energy of the ground state.

ii) As a warm up, find the ground state for the weaker potential

$$V(x) = \left(1 - \frac{2}{1 + \tanh^2(\lambda x)}\right). \tag{5}$$

Remember that the ground state wavefunction of a symmetric potential should be an even function and that we are looking for a bound states (so $E_0 < 0$). Use the Mathematica code to find the values of E_0 with 4 significant figures.

iii) Let us consider now the potential in eq. 3. Use the Mathematica code to find the values of E_0, E_1 and E_2 with 4 significant figures. *hint: which of the wavefunctions are even, which ones are odd?*