PHY 401 - QUANTUM PHYSICS 2008

HOMEWORK 1 - SOLUTION

A. 1) $i^i = \left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2} \simeq 0.20788\ldots$

There is a subtlety here. $i = e^{i(\pi/2+2n\pi)}$, for any integer n. Thus $i^i = \left(e^{i(\pi/2+2\pi n)}\right)^i = e^{-(\pi/2+2\pi n)}$. The issue is similar to $\sqrt{-1}$. What is usually meant by $\sqrt{-1}$ is $\sqrt{-1}=i$, but both $i^2=-1$ and $(-i)^2=-1$. Similarly with logs. What is usually meant by $\ln(i)$ is $\ln(i)=i\pi/2$ but not only $e^{i\pi/2}=i$ but $e^{i(\pi/2+2\pi n)}=i$ for any integer n.

2) $e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i\,1 = i$

3) $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x) \Rightarrow \frac{d^2}{dx^2}\psi(x) + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$\Rightarrow \psi(x) = A\cos kx + B\sin kx$, with $k = \frac{\sqrt{2mE}}{\hbar}$.

Imposing the boundary conditions:

$1 = \psi(0) = A \Rightarrow A = 1$

$0 = \left.\frac{d}{dx}\psi(x)\right|_{x=0} = -kB \Rightarrow B=0$

$$\boxed{\psi(x) = \cos kx\,,\quad k=\sqrt{2mE}/\hbar}$$

B. 1) $1 = \int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = \int_{-\infty}^{\infty} dx \, |A|^2 e^{-2\alpha x^2} = |A|^2 \sqrt{\frac{\pi}{2\alpha}}$,

where we used $\int_{-\infty}^{\infty} dx \, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$. So $A = \left(\frac{2\alpha}{\pi}\right)^{1/4}$.

2) $\langle x \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \, x \, \psi(x) = \int_{-\infty}^{\infty} dx \, A^2 e^{-2\alpha x^2} x = 0$ (odd integrand)

- $\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) x^2 \psi(x) = \int_{-\infty}^{\infty} dx \, A^2 e^{-2\alpha x^2} x^2 = A^2 \frac{d}{d(-2\alpha)} \int_{-\infty}^{\infty} dx \, e^{-2\alpha x^2}$

$= A^2 \frac{d}{d(-2\alpha)} \sqrt{\frac{\pi}{2\alpha}} = -A^2 \sqrt{\pi} \left(-\frac{1}{2}\right) \frac{1}{(2\alpha)^{3/2}}$

$= \sqrt{\frac{2\alpha}{\pi}} \sqrt{\pi} \, \frac{1}{2} \frac{1}{(2\alpha)^{3/2}} = \frac{1}{2} \frac{1}{2\alpha} = \frac{1}{4\alpha}$

$\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{4\alpha}$

3) $\langle p \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x) = \int_{-\infty}^{\infty} dx \, A e^{-\alpha x^2} \left(-i\hbar \frac{\partial}{\partial x}\right) A e^{-\alpha x^2}$

$= -i\hbar \int_{-\infty}^{\infty} dx \, A^2 e^{-\alpha x^2} (-2\alpha x) e^{-\alpha x^2} = 0$ (odd integrand)

$\langle p^2 \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \psi(x) = -\hbar^2 A^2 \int_{-\infty}^{\infty} dx \, e^{-\alpha x^2} \frac{\partial^2}{\partial x^2} e^{-\alpha x^2}$

$\underset{\text{(integration by parts)}}{=} \hbar^2 A^2 \int_{-\infty}^{\infty} dx \left(\frac{\partial}{\partial x} e^{-\alpha x^2}\right)^2 = \hbar^2 A^2 \int_{-\infty}^{\infty} dx \left(-2\alpha x \, e^{-\alpha x^2}\right)^2$

$= (-2\alpha)^2 \hbar^2 A^2 \int_{-\infty}^{\infty} dx \, x^2 e^{-2\alpha x^2} = (-2\alpha)^2 \hbar^2 A^2 \frac{d}{d(-2\alpha)} \int_{-\infty}^{\infty} dx \, e^{-2\alpha x^2}$

$= -4\alpha^2 \hbar^2 A^2 \frac{d}{d(2\alpha)} \sqrt{\frac{\pi}{2\alpha}} = -4\alpha^2 \hbar^2 A^2 \sqrt{\pi} \left(-\frac{1}{2}\right) \frac{1}{(2\alpha)^{3/2}}$

$= 4\alpha^2 \hbar^2 \sqrt{\frac{2\alpha}{\pi}} \sqrt{\pi} \, \frac{1}{2} \frac{1}{(2\alpha)^{3/2}} = \hbar^2 \alpha$

4) σ_x^2 is proportional to $1/\alpha$ while σ_p^2 is proportional to α. Making σ_x^2 smaller by increasing α makes σ_p^2 larger.

$$\sigma_x \sigma_p = \frac{1}{2\sqrt{\alpha}} \hbar \sqrt{\alpha} = \frac{\hbar}{2}$$

c. $$\frac{d}{dt}\langle p \rangle = \frac{d}{dt}\int_{-\infty}^{\infty} dx\, \Psi^*_{(x,t)} \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi_{(x,t)}$$

$$= -i\hbar \int_{-\infty}^{\infty} dx \left[\underbrace{\frac{\partial}{\partial t}\Psi^*}_{-\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi^* + \frac{i}{\hbar} V \Psi^*} \frac{\partial}{\partial x}\Psi + \Psi^* \frac{\partial}{\partial x} \underbrace{\frac{\partial}{\partial t}\Psi}_{\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi - \frac{i}{\hbar} V \Psi} \right]$$

$$= -i\hbar \int_{-\infty}^{\infty} dx \left[-\frac{i\hbar}{2m} \frac{\partial^2}{\partial x^2}\Psi^* \frac{\partial}{\partial x}\Psi + \frac{i}{\hbar} V \Psi^* \frac{\partial}{\partial x}\Psi + \frac{i\hbar}{2m} \Psi^* \frac{\partial^3}{\partial x^3}\Psi - \frac{i}{\hbar}\Psi^* \frac{\partial}{\partial x}(V\Psi) \right]$$

$$= -i\hbar \int_{-\infty}^{\infty} dx \left[-\frac{i\hbar}{2m} \Psi^* \frac{\partial^3}{\partial x^3}\Psi + \frac{i\hbar}{2m} \Psi^* \frac{\partial^3}{\partial x^3}\Psi + \frac{i}{\hbar} \Psi^* V \frac{\partial}{\partial x}\Psi - \frac{i}{\hbar}\Psi^* \frac{\partial V}{\partial x} \Psi - \frac{i}{\hbar} \Psi^* V \frac{\partial}{\partial x}\Psi \right]$$

$$= \int_{-\infty}^{\infty} dx\, \Psi_{(x,t)} \underbrace{\left(-\frac{\partial}{\partial x} V(x) \right)}_{F(x) = \text{force}} \Psi_{(x,t)} = \langle F \rangle$$

The equation above says that "F = ma" is valid in average. For large enough objects, where the average values are a good estimate of the outcome of every measurement, the ~~[illegible]~~ brackets can be dropped and we are left with Newton's law.