PHY 401 - QUANTUM PHYSICS

FINAL - SOLUTION

I. i) $$(\hat{H})^\dagger = \left(E_1 |a\rangle\langle b| + E_1 |b\rangle\langle a| + E_2 |c\rangle\langle c| \right)^\dagger$$

$$= E_1 |b\rangle\langle a| + E_1 |a\rangle\langle b| + E_2 |c\rangle\langle c|$$

$$= \hat{H}$$

ii) $$\hat{H}|1\rangle = \left[E_1 \left(|a\rangle\langle b| + |b\rangle\langle a| \right) + E_2 |c\rangle\langle c| \right] \left(\frac{|a\rangle + |b\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(E_1 |b\rangle + E_1 |a\rangle \right) = E_1 \left(\frac{|a\rangle + |b\rangle}{\sqrt{2}} \right)$$

$$= E_1 |1\rangle$$

$$\hat{H}|2\rangle = \left[E_1 |a\rangle\langle b| + E_1 |b\rangle\langle a| + E_2 |c\rangle\langle c| \right] \left(\frac{|a\rangle - |b\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(E_1 |b\rangle - E_1 |a\rangle \right) = -E_1 \left(\frac{|a\rangle - |b\rangle}{\sqrt{2}} \right)$$

$$= -E_1 |2\rangle$$

$$\hat{H}|3\rangle = \left[E_1 |a\rangle\langle b| + E_1 |b\rangle\langle a| + E_2 |c\rangle\langle c| \right] |c\rangle$$

$$= E_2 |c\rangle$$

$$= E_2 |3\rangle$$

$$\boxed{\begin{array}{ll} \text{eigenvalues:} & |1\rangle \to E_1 \\ & |2\rangle \to -E_1 \\ & |3\rangle \to E_2 \end{array}}$$

iii) If the value $E = E_1$ is found the system collapses to $|1\rangle = \frac{|a\rangle + |b\rangle}{\sqrt{2}}$ immediately after the measurement.

iv) The general sol. of the time-dependent Schrödinger eq. is.

$$|\psi(t)\rangle = c_1 e^{-iE_1t/\hbar}|1\rangle + c_2 e^{iE_1t/\hbar}|2\rangle + c_3 e^{-iE_2t/\hbar}|3\rangle$$

The initial condition $|\psi(T)\rangle = |a\rangle$ fixes the constants c_1, c_2 and c_3:

$$|\psi(T)\rangle = c_1 e^{-iE_1T/\hbar}|1\rangle + c_2 e^{iE_1T/\hbar}|2\rangle + c_3 e^{-iE_2T/\hbar}|3\rangle = |a\rangle$$

$$\Rightarrow \quad c_1 e^{-iE_1T/\hbar} = \langle 1|a\rangle = 1/\sqrt{2} \Rightarrow c_1 = \frac{e^{iE_1T/\hbar}}{\sqrt{2}}$$

$$c_2 e^{iE_1T/\hbar} = \langle 2|a\rangle = 1/\sqrt{2} \Rightarrow c_2 = \frac{e^{-iE_1T/\hbar}}{\sqrt{2}}$$

$$c_3 e^{-iE_2T/\hbar} = \langle 3|a\rangle = 0 \Rightarrow c_3 = 0$$

Thus

$$|\psi(t)\rangle = e^{-iE_1(t-T)/\hbar}|1\rangle + e^{iE_1(t-T)/\hbar}|2\rangle$$

$$= \frac{e^{-iE_1(t-T)/\hbar} + e^{iE_1(t-T)/\hbar}}{\sqrt{2}}|a\rangle + \frac{e^{-iE_1(t-T)/\hbar} - e^{iE_1(t-T)/\hbar}}{\sqrt{2}}|b\rangle$$

$$\boxed{|\psi(t)\rangle = \sqrt{2}\cos\frac{(t-T)E_1}{\hbar}|a\rangle + \sqrt{2}\, i \sin\frac{(t-T)E_1}{\hbar}|b\rangle}$$

v) $|\psi(T)\rangle = |a\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle + 0\ |3\rangle$

energy eigenstates

$E = E_1$ w/ probability $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$E = -E_1$ w/ probability $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$E = E_2$ w/ probability $0^2 = 0$

II. i) $\langle r\rangle = \int d^3r\ \varphi^*_{100}(\vec{r})\ |\vec{r}|\ \varphi_{100}(\vec{r})$

$$= \underbrace{4\pi}_{\text{integral over angles}} \int_0^\infty dr\ r^2 \left(\frac{2}{a_0^{3/2}}\right)^2 \left(\frac{1}{\sqrt{4\pi}}\right)^2 e^{-2r/a_0}\ r$$

$$= \frac{4}{a_0^3} \int_0^\infty dr\ e^{-2r/a_0}\ r^3 = \frac{4}{a_0^3}\ \frac{3!}{(2/a_0)^4} = \boxed{\frac{3}{2}\ a_0}$$

ii) $\underbrace{\psi(r)}_{\sqrt{\frac{\alpha^3}{\pi}}\, e^{-\alpha r}} = \sum_{nlm} c_{nlm} \underbrace{\varphi_{nlm}(\vec{r})}_{\text{energy eigenfunctions}}$

probability of $E_1 = |c_{100}|^2 = \left| \int d^3r \sqrt{\frac{\alpha^3}{\pi}}\, e^{-\alpha r}\ \underbrace{\varphi_{100}(r,\theta,\phi)}_{\text{ground state}} \right|^2$

$$= 4\pi \int_0^\infty dr\ r^2 \sqrt{\frac{\alpha^3}{\pi}}\ e^{-\alpha r}\ \frac{1}{\sqrt{4\pi}}\ \frac{2}{a_0^{3/2}}\ e^{-r/a_0}$$

$$= \frac{\sqrt{4\pi}\ 2}{\sqrt{\pi}} \left(\frac{\alpha}{a_0}\right)^{3/2} \frac{2!}{\left(\alpha + \frac{1}{a_0}\right)^3} = \boxed{8 \left(\frac{\alpha}{a_0}\right)^{3/2} \frac{1}{\left(\alpha + 1/a_0\right)^3}}$$

iii) a) $\psi = \frac{1}{6}\left[4\psi_{100} + 3\psi_{211} - \psi_{210} + \sqrt{10}\,\psi_{21-1}\right]$

$$\langle \hat{H} \rangle = \frac{1}{36}\left(4\langle\psi_{100}| + 3\langle\psi_{211}| - \langle\psi_{210}| + \sqrt{10}\langle\psi_{21-1}|\right)$$

$$\hat{H}\left(4|\psi_{100}\rangle + 3|\psi_{211}\rangle - |\psi_{210}\rangle + \sqrt{10}\,|\psi_{21-1}\rangle\right)$$

$$= \frac{1}{36}\left(4\langle\psi_{100}| + 3\langle\psi_{211}| - \langle\psi_{210}| + \sqrt{10}\langle\psi_{21-1}|\right)$$

$$\left(4E_1|\psi_{100}\rangle + 3E_2|\psi_{211}\rangle - E_2|\psi_{210}\rangle + \sqrt{10}\,E_2|\psi_{21-1}\rangle\right)$$

$$= \frac{1}{36}\left(16E_1 + 9E_2 + E_2 + 10E_2\right)$$

$$= \frac{16E_1 + 20E_2}{36} = \boxed{\frac{4E_1 + 5E_2}{9}}$$

b) $$\langle \hat{L}^2 \rangle = \frac{1}{36}\left(4\langle\psi_{100}| + 3\langle\psi_{211}| - \langle\psi_{210}| + \sqrt{10}\langle\psi_{21-1}|\right)$$

$$\hat{L}^2\left(4|\psi_{100}\rangle + 3|\psi_{211}\rangle - |\psi_{210}\rangle + \sqrt{10}\,|\psi_{21-1}\rangle\right)$$

$$= \frac{1}{36}\left(4\langle\psi_{100}| + 3\langle\psi_{211}| - \langle\psi_{210}| + \sqrt{10}\langle\psi_{21-1}|\right)$$

$$\left(4\cdot 0\,|\psi_{100}\rangle + 3\hbar^2\underbrace{l(l+1)}_{2} - \hbar^2\underbrace{l(l+1)}_{2}|\psi_{210}\rangle + \sqrt{10}\,\hbar^2\underbrace{l(l+1)}_{2}\right)$$

$$= \frac{1}{36}\left(18\hbar^2 + 2\hbar^2 + 20\hbar^2\right)$$

$$= \frac{40}{36}\hbar^2 = \boxed{\frac{10}{9}\hbar^2}$$