QUANTUM PHYSICS I

FINAL

I. THREE-STATE SYSTEM

A three-state system has a three-dimensional Hilbert space spanned by the orthonormal basis $\{|a\rangle, |b\rangle, |c\rangle\}$. Its hamiltonian is

$$\hat{H} = E_1 \left(|a\rangle\langle b| + |a\rangle\langle b|\right) + E_2|c\rangle\langle c|, \tag{1}$$

where $E_1 > E_2$ are real, positive constants.

i) Show that the hamiltonian above is hermitian.

ii) Show that $|1\rangle = (|a\rangle + |b\rangle)/\sqrt{2}$, $|2\rangle = (|a\rangle - |b\rangle)/\sqrt{2}$ and $|3\rangle = |c\rangle$ are eigenstates of the hamiltonian. What are their corresponding eigenvalues?

iii) At time $t = 0$ the energy is measured and the value $E = E_1$ is found. What is the state of the system immediately after the measurement?

iv) At time $t = T$ the system is prepared to be at $|\psi(T)\rangle = |a\rangle$. What is the state of the system at a time $t > T$.

v) At time $t = T$ the system is prepared to be at $|\psi(T)\rangle = |a\rangle$, just like in the previous question. If the energy is measured at a time $t > T$, what are the possible outcomes and what are their probabilities?

II. HYDROGEN ATOM

i) Calculate the expectation value ($\langle r\rangle$) of the distance between the electron and the nucleus for an electron in the ground state of a hydrogen atom.

ii) An electron is found orbiting a proton with the (already normalized) wave function

$$\psi(\mathbf{r}) = \sqrt{\frac{\alpha^3}{\pi}} e^{-\alpha r}. \tag{2}$$

If an energy measurement is made, what is the probability of finding the ground state energy?

iii) An electron in the Coulomb field of a proton is in a state described by the wave function

$$\psi(\mathbf{r}) = \frac{1}{6}\left[4\psi_{100}(\mathbf{r}) + 3\psi_{211}(\mathbf{r}) - \psi_{210}(\mathbf{r}) + \sqrt{10}\psi_{21-1}(\mathbf{r})\right], \tag{3}$$

where $\psi_{nlm}(\mathbf{r})$ are the energy eigenfunctions with quantum numbers n, l and m.

a) What is the expectation value of the energy?

b) What is the expectation value of $\hat{\mathbf{L}}^2$?

USEFUL FORMULAE

Radial wave function of the hydrogen atom

$$\begin{aligned} R_{10}(r) &= \frac{2}{a_0^{3/2}} e^{-r/a_0}, \\ R_{20}(r) &= \frac{2}{(2a_0)^{3/2}} \left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}, \\ R_{21}(r) &= \frac{1}{\sqrt{3}} \frac{1}{(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/(2a_0)}, \end{aligned} \tag{4}$$

where $a_0 = 4\pi\epsilon_0\hbar^2/(me^2)$ is the Bohr radius.

Spherical Harmonics

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}}, \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta, \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\ Y_{22} &= \sqrt{\frac{15}{32\pi}} e^{2i\phi} \sin^2\theta, \\ Y_{21} &= -\sqrt{\frac{15}{8\pi}} e^{i\phi} \sin\theta\cos\theta, \\ Y_{20} &= \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1). \end{aligned} \tag{5}$$

Integrals

$$\int_0^\infty dx\ e^{-Ax} x^n = \frac{n!}{A^{n+1}}, \tag{6}$$