Basic differential equations (for before the beginning of class)

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Not a substitute for a differential equations class, obviously. Just the absolute minimal the students of PHY401 should know *before* the class starts.

Everybody is familiar with algebraic equations like

$$y^2 - y - 2 = 0,$$

or

$$y^3 + \frac{1}{y} = 57. \tag{1}$$

To solve an equation like 1 means to find the values of y that, when plugged in (1) makes the equation true. For instance, in the first example above, $y = 2$ and $y = -1$ are solutions. In general, there may be one solution to an equation, or two, or 17, or an infinite number. Or maybe there is none. It is very useful to know how many solutions a given equation has even if we don't know how to solve it. For instance, in the case of second degree polynomial equations as the first one in (1) we know that there may be 0, 1 or 2 solutions if we restrict ourselves to real numbers but that there are always two (and only two) solutions if we allow for complex number solutions[1]. Notice that solving an algebraic equation can be very difficult but verifying that a given solution is correct is usually very simple. It is just a matter of taking the presumed solution, plug it back in the equation and see whether it works. So *guessing solutions and checking if they work is a perfectly rigorous, and sometimes efficient, way of solving equations.* Neophytes seem to have, for some obscure reason, a hard time with the guessing-and-verifying method. Don't be one of them.

Differential equations are like algebraic equation except that, instead of the unknown being a number, it is a function. For instance:

$$\frac{df(t)}{dt} = f^2(t),$$
$$\frac{d^2y(x)}{dx^2} = Ay(x),$$
$$\frac{d^2y(x)}{dx^2} = \sin(\omega x). \tag{2}$$

Just like algebraic equations, differential equations may have one unique solution, or many, or none. It turns out that is easy to know how many solutions a given differential equation has. Better yet, there is a simple, intuitive picture that tell us how many solutions there are. After we understand this intuitive argument you should never, ever again wonder whether we have all solutions of a given equation. I'll relegate this argument to the appendix though and now will just state the result

A differential equation involving up to the n^{th} derivative has a n parameter family of solutions and we need to know n additional conditions to specify one unique solution.

The solutions of most differential equations can not be written in terms of simple functions we have names for like powers, sines, exponentials, ... Even if the solutions can be written in terms of simple functions it may be very hard to find them. A few very simple equations appear over and over again so it's essential to know their solutions immediately. This is like knowing the multiplication table: there is nothing deep about it but, if you don't know it by heart, it stops us from understanding deep things. The list of the essential equations and their solutions is:

$$\frac{dy(t)}{dt} = A \Rightarrow y(t) = A\ t + C. \tag{3}$$

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[1] You know that from High School, right ?

$$\frac{dy(t)}{dt} = Ay(t) \Rightarrow y(t) = Ce^{At}. \tag{4}$$

$$\begin{aligned}
\frac{d^2y(t)}{dt^2} + Ay(t) = 0 \Rightarrow y(t) &= C_1 \sin(\sqrt{A}t) + C_2 \cos(\sqrt{A}t) \\
&= C_1' \sin(\sqrt{A}(t - t_1)) = C_2' \cos(\sqrt{A}(t - t_2)) \\
&= C_3\ e^{i\sqrt{A}t} + C_4\ e^{-i\sqrt{A}t} \\
&= C_5 \sinh(\sqrt{-A}t) + C_6 \cosh(\sqrt{-A}t) \\
&= C_7\ e^{\sqrt{-A}t} + C_8\ e^{-\sqrt{-A}t} \\
&= C_7' \sinh(\sqrt{-A}(t - t_7)) = C_8' \cosh(\sqrt{-A}(t - t_8))
\end{aligned} \tag{5}$$

Notice that each one of these solutions depend on two parameters (C_1 and C_2 or C_1' and t_1, ...). Where do these solutions come from ? *It does not matter !* You can think that the muses whispered the solutions in my ears. What matters is that, given these solutions, you can verify for yourself that they are, in fact, solutions[2]. And since they depend on as many arbitrary constants (one for the two first equations, two for the last) as the order of the equations, we know that they are the most general solution for the equations.

A natural question is: among the many ways of writing the solutions in eq. , which one should be used in a particular problem? That depends but, in general, the rule is to write the solution in terms of real exponentials or hyperbolic functions if $A > 0$ and complex exponentials or trigonometric functions if $A < 0$.

The constants $C, C_1, \cdots$ are parameterizing the family of solutions we mentioned above. For instance, $y(t) = At+C$ is a one-parameter family of solutions because for every value of C we have a solution of the $dy/dt = A$ equation. A unique solution is specified when an additional condition is added, fixing the value of C. For instance, the solution of

$$\frac{dy(t)}{dt} = 3y(t) \tag{6}$$

with the initial condition $y(2) = 4$ can be found as

$$y(t) = Ce^{3t} \Rightarrow y(2) = Ce^6 = 4 \Rightarrow C = 4e^{-6} \approx 0.01 \Rightarrow y(t) = 4\ e^{3t-6}. \tag{7}$$

Notice that the general solution of (5) can be written in multiple ways. The two first ones are convenient if $A > 0$, the last two if $A < 0$. As an example, let us take the boundary problem:

$$\begin{aligned}
\frac{d^2y(t)}{dt^2} + 4y(t) = 0, \\
y(0) = 1,\ \frac{dy(0)}{dt} = 0.
\end{aligned} \tag{8}$$

Since $A = 4$ is positive, it is more convenient to use one of the two top form in (5). If we write the solution in terms of sines and cosines we can determine C_1 and C_2 as

$$\begin{aligned}
1 &= y(0) = C_1\ 0 + C_2\ 1, \\
0 &= y'(0) = 2C_1\ 1 - 2C_2\ 0
\end{aligned} \tag{9}$$

so $C_1 = 0$, $C_2 = 1$ and $y(t) = \cos(2t)$. Of course, we would find the same solution had we chosen the second line in (5). In that case $C_3 = C_4 = \frac{1}{2}$.

A. Appendix: n^{th} order equations have a n parameter family of solutions

For the sake of the argument, take an equation like

$$\frac{dy(t)}{dt} = 2y(t). \tag{10}$$

[2] Do it !

We can think of this equation as telling us that the rate of change of $y(t)$ with "time" t is given by the value of $y(t)$ itself. The equation tell us how $y(t)$ as t changes, but does not tell us what the initial value of $y(t)$ is. There is one solution for each value of this initial condition, in other words, the set of all solutions forms a one-parameter family of functions. To be more precise, let us approximate the derivative by

$$\frac{dy(t)}{dt} \approx \frac{y(t+\Delta t) - y(t)}{\Delta t}, \tag{11}$$

where Δt is a "small" number. Equation (10) can be written as

$$y(t+\Delta t) = y(t) + 2\Delta t \; y(t). \tag{12}$$

Now, suppose we know $y(t)$ for some initial "time", say , we know that $y(17) = 1$. We can use (12) to calculate the value $y(17+\Delta t)$. But then we can use the same equation again, now with $t = 17+\Delta t$ and find the value of $y(17+2\Delta t)$. If we keep doing this, we can find the value of $y(t)$ at any value of t ! In fact, that's one (of the most simple) ways that differential equations are solved numerically, since this process is is easily implemented on a computer. The result of this procedure is not an analytic expression like $y(t) = \sin(t^2)$ or $y(t) = e^{2t}$ but that is irrelevant. What matters is that we know the value of $y(t)$ for every t. Now, from the way we solved the equation (10) one thing is clear: had we started with a different value for $y(t = 17)$ we would have found a different solution. But, with a given fixed value of the initial condition $y(t = 17)$, the solution is unique and given by the procedure above. In other words, the solutions of (10) form a one-parameter set of functions. Actually, the general solution of (10) [3]

$$y(t) = y(0)e^{2t} \tag{13}$$

shows explicitly that for every value of $y(0)$ we have a different solution. This result generalizes to any first-order differential equation, namely, any equation with first, but no second, ... derivatives of the unknown $y(t)$.

How does this result generalizes to second order equations like, for instance,

$$\frac{d^2y(t)}{dt^2} = 5 \; y(t) \; ? \tag{14}$$

The approximation of the second derivative is

$$\frac{d^2y(t)}{dt^2} \approx \frac{1}{\Delta t}\left(\frac{y(t+\Delta t) - y(t)}{\Delta t} - \frac{y(t) - y(t-\Delta t)}{\Delta t}\right) = \frac{y(t+\Delta t) + y(t-\Delta t) - 2y(t)}{\Delta t^2}. \tag{15}$$

The analogue of (12) will know contain not only $y(t + \Delta t)$ and $y(t)$ but also $y(t - \Delta t)$. The analogue of (12) is

$$y(t+\Delta t) = (5\Delta t^2 + 2)y(t) - y(t-\Delta t). \tag{16}$$

In order to find the value of the solution at time $t + \Delta t$ we need to know $y(t)$ **and** $y(t - \Delta t)$. We get a different solution for every value of these two parameters. These two parameters can be chosen to be the value of $y(t)$ at two different times $y(t_1)$ and $y(t_2)$, or the value of y and dy/dt at some The set of solution of (14) is then a two parameter family. In fact, the general solution of (14) is

$$y(t) = \frac{dy(0)}{dt}\frac{\sin(\sqrt{5}t)}{\sqrt{5}} + y(0)\cos(\sqrt{5}t). \tag{17}$$

The general rule is then: an equation with up to the n^{th} derivative has a n apramenter family of solutions. In order to specify one unique solution we need n additional conditions (sometimes called "boundary" or "initial" conditions.[4]:

[3] It is very instructive to solve the equation (10) numerically and compare the answer with the exact analytic expression. Take the initial time to be $t = 0$, $y(0) = 1$ and $\Delta t = 0.1$.

[4] If you are mathematically oriented you may want to find out the precise conditios for this to be true.