Complex numbers(for before the beginning of class)

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Basic complex algebra. Just the absolute minimal the students of PHY401 should know *before* the class starts.

I. INTRODUCTION

There are two good ways of thinking about complex numbers. One is algebraic. We can think of a complex number as a pair of real numbers: $z = (x, y)$ where the operation of sum and multiplication are defined by

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2). \end{aligned} \tag{1}$$

The first real number is called the real part and the second one the imaginary part of the complex number z. The real numbers can be thought as a subset of the complex numbers, namely , the set of complex numbers with a vanishing imaginary part like $z = (x, 0)$. You can verify that all the properties of sums and multiplications of real numbers (like $z_1 z_2 = z_2 z_1, \cdots$) are also satisfied by the operations above. The only thing that is different is that there are complex numbers, like "$i = (0, 1)$", whose square is a negative number: $i^2 = ii = (0, 1)(0, 1) = (-1, 0)$ (remember that we identified $(-1, 0)$ with he real number -1). Nobody writes complex numbers like (x, y) and uses that complicated multiplication rule in (1) though. The standard notation is $z = x + iy$. This makes it really easy to do calculations with complex numbers; instead of remembering the strange multiplication rule in eq.(1) all we have to remember is that $i^2 = -1$ and then use the same algebra rules as real numbers that we learned in kindergarden. For instance:

$$\begin{aligned} (2 - 3i) + (7 + 5i) &= 9 + 2i, \\ (2 - 3i) * (7 + 5i) &= 2 * 7 + 2 * 5i - 3i * 7 - 3i * 5i = 14 + 10i - 21i + 15 = 29 - 11i, \end{aligned} \tag{2}$$

There is also a new operation that is very common: *complex conjugation*. It is denoted by a little star like in z^* and is defined by $z^* = (x + iy)^* = x - iy$, that is, it flips the sign of the imaginary part. Look at some manipulation involving the complex conjugation:

$$\begin{aligned} (4 - 2i)^* &= 4 + 2i, \\ (4 + 3i)(4 + 3i)^* &= (4 + 3i)(4 - 3i) = 4^2 - (3i)^2 = 16 + 9 = 25, \\ (x + iy)(x + iy)^* &= x^2 + y^2, \\ \frac{1}{3 + 2i} &= \frac{1}{3 + 2i}\frac{(3 + 2i)^*}{(3 + 2i)^*} = \frac{3 - 2i}{9 + 4} = \frac{3}{13} - i\frac{2}{13}. \end{aligned} \tag{3}$$

Exercise: Calculate

$$\begin{aligned} &\frac{1}{-1 + 5i} \\ &\frac{2 - i}{3 + 4i} \\ &(1 + i)^2 \\ &(x + iy) + (x + iy)^*, \quad \text{(assume x, y are real)} \end{aligned}$$

We can also think of the real and imaginary parts of a complex number as coordinates of a point on a plane. The good old real numbers lie on the horizontal axis, the purely imaginary numbers on the vertical axis. it is probably a good idea to visualize the complex plane in your mind every time there is a complex number in class.

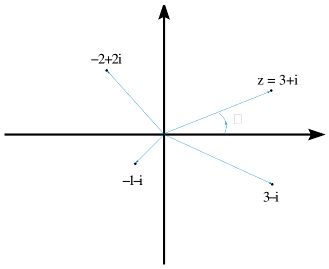

FIG. 1: Complex plane and a few complex numbers

The position vector of a point on the plane can also be specified by its length and direction. What are the length and direction of the complex number $z = x + iy$? The length (or *magnitude* or *modulus*) is usually denoted by $|z|$ is given by the Pythagorean theorem

$$|z| = \sqrt{x^2 + y^2} = \sqrt{zz^*}. \tag{4}$$

If you remember your triangles from high school you can see that the angle θ between the vector representing a complex number and the horizontal axis is given by

$$\tan\theta = \frac{y}{x}. \tag{5}$$

This angle is called the "argument" or "phase" of the complex number.

The sum of two complex numbers is determined by (look at eq.(1)) adding up each component so complex numbers add just like vectors do, as show in figure 1.

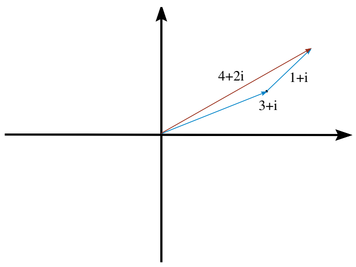

FIG. 2: Complex numbers add like two-dimensional vectors do.

What about multiplication? We could use the definition in eq.(1) together with eqs.(4) and (5) to prove that the magnitudes are multiplied and the angles are summed:

$$z_3 = z_1 z_2 \Rightarrow \begin{array}{c} |z_3| = |z_1||z_2| \\ \theta_3 = \theta_1 + \theta_2 \end{array} \tag{6}$$

We will not prove eq.(6) now because there is a much slicker way of doing this later. But the result is useful. For instance, multiplying a complex number by a real number only changes its magnitude, not its direction. Multiplying a a complex number by a number with magnitude 1 only shifts the phase.

We know how to sum, subtract, multiply and divide complex numbers (division is just multiplication by $1/z$ and we have already done that in the examples). Can we do more complicated things like taking the cosine of a complex number ? The way to extend function of real numbers to the complex numbers is to think about their Taylor series. For instance, we know (or should know) that

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots . \tag{7}$$

But we know how to compute powers of complex numbers so we can also compute the cosine of a complex number.

For instance:

$$\begin{aligned}\cos(i) &= 1 - \frac{1}{2}i^2 + \frac{1}{4!}i^4 + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{4!} + \cdots \\ &= \frac{e + e^{-1}}{2}. \end{aligned} \tag{8}$$

(Again, in the last step I used something I'll show only later. Sorry.) Now, we are ready to derive the central result of these notes and, arguably, the most beautiful result of all Mathematics. What is the exponential of a purely imaginary number? Well, we use the exponential Taylor series as a guide:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \tag{9}$$

thus

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots \\ &= 1 + i\theta - \frac{1}{2!}(\theta)^2 - i\frac{1}{3!}(\theta)^3 + \cdots \\ &= 1 - \frac{1}{2!}\theta^2 + \cdots + i(\theta - \frac{1}{3!}\theta^3 + \cdots) \\ &= \cos\theta + i\sin\theta. \end{aligned} \tag{10}$$

This result is called the Euler formula and it deserves a box around it.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.} \tag{11}$$

That is, the real and imaginary parts of $e^{i\theta}$ are the cosine and sine of θ. Its magnitude is 1 (since $\sin^2\theta + \cos^2\theta = 1$) and its phase is θ (since $\tan\theta = \textit{real part/imaginary part}$). Multiplying $e^{i\theta}$ by a positive real number A we get $Ae^{i\theta}$ that is a complex number with magnitude A and phase θ.

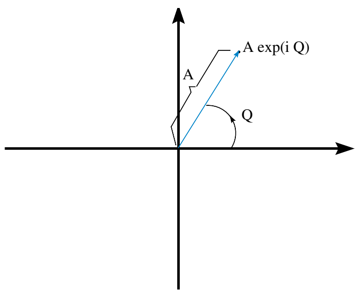

FIG. 3: $Ae^{i\theta}$ has magnitude A and angle θ. (The "Q" stands for the θ).) *Every time you come across a complex number you should think of this figure.* A complex number is a point in a plane with a magnitude (distance from the center) and an argument (angle with the horizontal axis).

Notice that writing $z = x + iy$ is like using cartesian coordinates and writing $z = Ae^{i\theta}$ is like using polar coordinates. We can go from one to the other very easily

$$\begin{aligned} A &= \sqrt{x^2 + y^2}, \\ \tan\theta &= \frac{y}{x}. \end{aligned} \tag{12}$$

We can find all sorts of amazing, useful, curious or downright weird results using the Euler formula. I'll list a few here so you can get practice on complex numbers proving them (sometimes the first one is called the "Euler formula")

:

$$
\begin{aligned}
e^{i\pi}+1 &= 0,\\
e^{i2\pi} &= 1,\\
e^{i4\pi} &= 1,\\
e^{-i2\pi} &= 1,\\
i^i &= e^{-\frac{\pi}{2}} \approx 0.2078...,\\
\sqrt{i} &= \frac{1+i}{\sqrt{2}},\\
\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2},\\
\sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}.
\end{aligned}
\tag{13}
$$

These last two relations allow us to never ever again have to remember those annoying trigonometric identities we learned in High School. Using complex numbers we can derive them on the fly. For instance, if $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$

$$
(\cos(\theta_1)+i\sin(\theta_1))\,(\cos(\theta_2)+i\sin(\theta_2)) = z_1 z_2 = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2) \tag{14}
$$

Equating the real and imaginary parts we find

$$
\begin{aligned}
\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2,\\
\sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2.
\end{aligned}
\tag{15}
$$

We can also prove something i mentioned before without proof, namely, that in multiplying two complex numbers their magnitude multiply while their phase add. In fact, if $z_1 = A_1e^{i\theta_1}$ and $z_2 = A_2e^{i\theta_2}$, $z = Ae^{i\theta} = z_1z_2 = A_1A_2e^{i(\theta_1+\theta_2)}$, so $A = A_1a_2$ and $\theta = \theta_1+\theta_2$.

At this point I suggest you stop reading books and try manipulating complex numbers for yourself. If you don't know what to do, look at the exercises in "Mathematical Tools for Physics", by James Nearing, freely available at `http://www.physics.miami.edu/∼ nearing/mathmethods/`.

Finally, I have to answer the question that is (or should be) or your mind all along: what is the point of complex numbers ? What are they good for? Physical quantities like position, velocity, energy, ... are always real, right ? There are several answers for this question. The first is that complex numbers help you do things with the good old real numbers that would be difficult or impossible without them. One example was the derivation of the trigonometric identities above. Another is the computation of, for instance

$$
\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1} = \frac{\pi}{e}. \tag{16}
$$

In Quantum Mechanics, though, the relevance of complex numbers is more direct. As we will see soon, the most basic object of Quantum Mechanics, the "wave function", is actually a complex function. So, as long as you are a physicist, complex numbers will be in your life.

There is another natural question frequently asked. If complex numbers can be thought as two-dimensional vectors, is the some kind of hypercomplex numbers that are like three-dimensional vectors? The answer to that depends on what exactly you want to call a "number". There is no three-dimensional numbers satisfying the usual rules that real and complex numbers obey (for the mathematically oriented-the axiom of a "field"). But if you are willing to have a multiplication rule that is not commutative (so $z_1z_2 \neq z_2z_1$) there is a *four* dimensional generalization of complex numbers called *quaternions*. If you also give up the associative rule, there is the eight-dimensional algebra of *octonions*. And that is it. No more[1]. Of course, if you start calling all kinds of different algebraic structures "numbers", there are tons and tons of generalizations of the concept of real numbers. Some of them will even appear in our Quantum Mechanics class.

[1] the precise statement is that there are only four normed division algebras.