MIDTERM II
PHY 401

① a) $\int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = 1 \Rightarrow \int_{-L}^{L} dx \, |A|^2 = 2L|A|^2 = 1 \Rightarrow A = \frac{1}{\sqrt{2L}}$

b) momentum distribution $= |c(p)|^2$ where $\psi(x) = \int_{-\infty}^{\infty} dp \; c(p) \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$

$$\Rightarrow c(p) = \int_{-\infty}^{\infty} dx \, \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x)$$

$$= \int_{-L}^{L} dx \, \frac{A}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} = \frac{A}{\sqrt{2\pi\hbar}} \frac{\hbar}{-ip} e^{-ipx/\hbar}\Big|_{-L}^{L}$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \frac{i\hbar}{p} \underbrace{\left(e^{-ipL/\hbar} - e^{ipL/\hbar}\right)}_{-2i \sin pL/\hbar}$$

$$= \frac{A \, 2\hbar}{\sqrt{2\pi\hbar}\, p} \sin pL/\hbar$$

$$|c(p)|^2 = \frac{1}{2L} \frac{4\hbar^2}{2\pi\hbar p^2} \sin^2 pL/\hbar = \frac{2L}{2\pi\hbar} \left(\frac{\sin pL/\hbar}{pL/\hbar} \right.$$

c) position distribution $= |\psi(x)|^2 = \begin{cases} 1/2L, & -L < x < L \\ 0, & |x| > L \end{cases}$

② a) $x<0$: $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x)+V_0\varphi(x)=E\varphi(x) \Rightarrow \varphi(x)=Ae^{ik_0x}+Be^{-ik_0x},$$ with $k_0=\frac{\sqrt{2m(E-V_0)}}{\hbar}$

$x>0$: $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x)=E\varphi(x) \Rightarrow \varphi(x)=Ce^{ikx}+De^{-ikx},$$ with $k=\sqrt{2mE}/\hbar$.

b) $$\varphi(0+\epsilon)=\varphi(0-\epsilon)$$
$$\frac{d}{dx}\varphi(x)\bigg|_{x=0+\epsilon}=\frac{d}{dx}\varphi(x)\bigg|_{x=0-\epsilon}$$

c) $$T=\left|\frac{J_{out}}{J_{in}}\right|=\frac{k\,|C|^2}{k_0\,|A|^2}$$

continuity @ $x=0$: $A+B=C+\underbrace{D}_{=0}$ (no particles coming back)

continuity of derivative @ $x=0$: $ik_0(A-B)=ik(C-\underbrace{D}_{=0})$

$\Downarrow$

$$A+B=C=\frac{k_0}{k}(A-B) \Rightarrow A\left(1-\frac{k_0}{k}\right)=B\left(-1-\frac{k_0}{k}\right)$$

$$\Rightarrow \frac{B}{A}=\frac{k_0-k}{k_0+k},\ \frac{C}{A}=1+\frac{B}{A}$$

$$T=\frac{k}{k_0}\left|1+\frac{k_0-k}{k_0+k}\right|^2=\frac{k}{k_0}\left(\frac{2k_0}{k_0+k}\right)^2=\left(\frac{2}{k_0+k}\right)^2 kk_0$$