QUANTUM PHYSICS I - FALL 2007
MIDTERM I - SOLUTIONS

A. a) $\frac{mv^2}{r} = m\omega^2 r$ (Newton's law) $\Rightarrow v = \omega r$

$L = mvr = n\hbar$ (Bohr's angular momentum quantization) $\Rightarrow v = \frac{n\hbar}{mr}$

Elliminating v we have $r = \sqrt{n\hbar/m\omega}$. The total energy is then

$$E = \underbrace{\frac{mv^2}{2}}_{\text{kinetic}} + \underbrace{\frac{m\omega^2 r^2}{2}}_{\text{potential}} = \frac{m\omega^2 r^2}{2} + \frac{m\omega^2 r^2}{2} = m\omega^2 r^2 = n\hbar\omega$$

$$\boxed{E_n = n\hbar\omega, \quad n = 0, 1, 2, \dots}$$

NOTE: THIS METHOD GIVES THE CORRECT ENERGY LEVELS AS $n \to \infty$. In particular, THE CORRECT GROUND STATE ENERGY ($n=0$) IS NOT ZERO.

b) The energy carried off by the photon is the difference of the energies of the initial and final states: $E_\gamma = E_1 - E_0 = \hbar\omega$. The photon frequency is $\nu = \frac{E_\gamma}{h} = \frac{\hbar\omega}{h} = 2\pi\omega$. It's only a coincidence that the frequency of the photon is the same as the frequency $2\pi\omega$ of the harmonic oscillator.

B. a) Each stationary state evolves as $\Psi_n(x,t) = e^{-iE_nt/\hbar}\,\varphi_n(x)$ so, due to the linearity of the Schrödinger eq. we have

$$\boxed{\Psi(x,t) = \sqrt{\frac{5}{7}}\; e^{-i\frac{\hbar\pi^2}{2ma^2}t}\,\varphi_1(x) + \sqrt{\frac{2}{7}}\; e^{-\frac{4\hbar\pi^2}{2ma^2}t}\,\varphi_2(x)}$$

b) Two possible outcomes:

$$E = E_1 = \frac{\pi^2\hbar^2}{2ma^2} \quad \text{w/ probability} \quad \sqrt{\tfrac{5}{7}}^{\,2} = \frac{5}{7}$$

$$E = E_2 = \frac{4\pi^2\hbar^2}{2ma^2} \quad \text{w/ probability} \quad \sqrt{\tfrac{2}{7}}^{\,2} = \frac{2}{7}$$

c) $$\langle E \rangle = \underset{\substack{\uparrow \\ \text{prob. of} \\ E_1}}{E_1 \frac{5}{7}} + \underset{\substack{\uparrow \\ \text{prob. of} \\ E_2}}{E_2 \frac{2}{7}} = \left[\frac{5}{7} + \frac{4\cdot 2}{7}\right]\frac{\pi^2\hbar^2}{2ma^2} = \frac{13}{7}\frac{\pi^2\hbar^2}{2ma^2}$$

Alternatively we can do

$$\langle E \rangle = \int_{-\infty}^{\infty} dx\ \psi^*(x,0)\ \hat{H}\ \psi(x,0) = \int_{-\infty}^{\infty} dx \left[\sqrt{\tfrac{5}{7}}\,\psi_1^*(x) + \sqrt{\tfrac{2}{7}}\,\psi_2^*(x)\right] \hat{H} \left[\sqrt{\tfrac{5}{7}}\,\psi_1(x) + \sqrt{\tfrac{2}{7}}\,\psi_2(x)\right]$$

$$= \int_{-\infty}^{\infty} dx \left[\sqrt{\tfrac{5}{7}}\,\psi_1^*(x) + \sqrt{\tfrac{2}{7}}\,\psi_2^*(x)\right]\left[\sqrt{\tfrac{5}{7}}\,E_1\,\psi_1(x) + \sqrt{\tfrac{2}{7}}\,E_2\,\psi_2(x)\right]$$

$$= \underbrace{\int_{-\infty}^{\infty} dx\ \frac{5}{7} E_1 |\psi_1(x)|^2}_{\frac{5}{7}E_1} + \underbrace{\int_{-\infty}^{\infty} dx\ \frac{2}{7} E_2 |\psi_2(x)|^2}_{\frac{2}{7}E_2} + \frac{\sqrt{10}}{7}\underbrace{\int_{-\infty}^{\infty} dx\ E_2\,\psi_1^*(x)\,\psi_2(x)}_{=0}$$

$$+ \frac{\sqrt{10}}{7}\underbrace{\int_{-\infty}^{\infty} dx\ E_1\,\psi_2^*(x)\,\psi_1(x)}_{=0}$$

$$= \frac{5}{7}E_1 + \frac{2}{7}E_2.$$

d) Similarly to a):

$$\boxed{\Psi(x,t) = \sqrt{\frac{5}{7}}\, e^{-\frac{i\hbar\pi^2 t}{2ma^2}}\, \varphi_1(x) - i\sqrt{\frac{2}{7}}\, e^{-\frac{i4\hbar\pi^2 t}{2ma^2}}\, \varphi_2(x)}$$

I'll compute $\langle E \rangle$ now the silly way

$$\langle E \rangle = \int_{-\infty}^{\infty} dx\, \Psi^*(x,0)\, \hat{H}\, \Psi(x,0) = \int_{-\infty}^{\infty} dx \left[\sqrt{\frac{5}{7}}\varphi_1^* + i\sqrt{\frac{2}{7}}\, \varphi_2^*(x)\right] \hat{H} \left[\sqrt{\frac{5}{7}}\, \varphi_1(x) - i\sqrt{\frac{2}{7}}\, \varphi_2(x)\right]$$

$$= \sqrt{\frac{5}{7}}^{\,2} E_1 + \left[i\sqrt{\frac{2}{7}}\right]\left[-i\sqrt{\frac{2}{7}}\right] E_2$$

$$= \frac{5}{7} E_1 + \frac{2}{7} E_2 .$$

C. a) $\displaystyle\int_{-\infty}^{\infty} dx\, |\varphi(x)|^2 = \int_{-\infty}^{\infty} dx\, |A|^2 e^{-2\alpha x^2} = |A|^2 \frac{\sqrt{\pi}}{\sqrt{2\alpha}} = 1$

$\Rightarrow A = \left(\frac{2\alpha}{\pi}\right)^{1/4}$ (multiplied by an arbitrary phase, like 1)

b) $$\langle x \rangle = \int_{-\infty}^{\infty} dx\, \varphi^*(x)\, \hat{x}\, \varphi(x) = \int_{-\infty}^{\infty} dx\, |A|^2 \underbrace{e^{-\alpha x^2}\, x\, e^{-\alpha x^2}}_{\text{odd integrand}} = 0$$

$$\langle p \rangle = \int_{-\infty}^{\infty} dx\, \varphi^*(x)\, \hat{p}\, \varphi(x) = \int_{-\infty}^{\infty} dx\, |A|^2 e^{-\alpha x^2} (-i\hbar) \frac{d}{dx} e^{-\alpha x^2}$$

$$= |A|^2 \int_{-\infty}^{\infty} dx\, \underbrace{(-i\hbar)(-2\alpha x)\, e^{-2\alpha x^2}}_{\text{odd integrand}} = 0$$

$$\langle E \rangle = \int_{-\infty}^{\infty} dx\, \varphi^*(x)\, \frac{\hat{p}^2}{2M}\, \varphi(x) = \int_{-\infty}^{\infty} dx\, |A|^2 e^{-\alpha x^2} \frac{-\hbar^2}{2M} \frac{d^2}{dx^2} e^{-\alpha x^2}$$

only kinetic, no potential

$$= \frac{|A|^2\hbar^2}{2M} \int_{-\infty}^{\infty} dx \left(\frac{d}{dx} e^{-\alpha x^2}\right)^2 = \frac{|A|^2\hbar^2}{2M} \int_{-\infty}^{\infty} dx\, (-2\alpha)^2 x^2 e^{-2\alpha x^2}$$

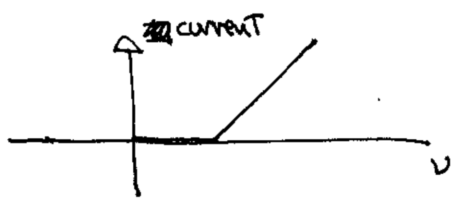
integration by parts

$$= \frac{|A|^2 \hbar^2}{2M} 4\alpha^2 \underbrace{\int_{-\infty}^{\infty} dx \; x^2 e^{-2\alpha x^2}}_{\frac{\sqrt{\pi}}{2(2\alpha)^{3/2}}} = \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{\hbar^2}{2M} 4\alpha^2 \frac{\sqrt{\pi}}{2(2\alpha)^{3/2}}$$

$$= \frac{\hbar^2 \alpha}{2M}$$

$$\boxed{\langle E \rangle = \frac{\hbar^2 \alpha}{2M}}$$

D. a)



b)