QUANTUM PHYSICS I - FALL 07

HOMEWORK 6 SOLUTION

1. In the $\{|a\rangle, |b\rangle, |c\rangle\}$ basis the hamiltonian matrix is:

$$\mathbb{H} = \begin{pmatrix} \langle a|\hat{H}|a\rangle & \langle a|\hat{H}|b\rangle & \langle a|\hat{H}|c\rangle \\ \langle b|\hat{H}|a\rangle & \langle b|\hat{H}|b\rangle & \langle b|\hat{H}|c\rangle \\ \langle c|\hat{H}|a\rangle & \langle c|\hat{H}|b\rangle & \langle c|\hat{H}|c\rangle \end{pmatrix} = \begin{pmatrix} E_0 & T & T \\ T & E_0 & T \\ T & T & E_0 \end{pmatrix}$$

The eigenvalues of $\hat{H}$ are given by the solutions of:

$$\hat{H}|\varphi\rangle = E|\varphi\rangle \Rightarrow \det\begin{pmatrix} E_0-E & T & T \\ T & E_0-E & T \\ T & T & E_0-E \end{pmatrix} = 0 \Rightarrow \boxed{E = E_0-T, E_0-T, E_0+2T}$$

degenerate ground state

2. If we write the eigenstates of $\hat{H}$ as ($|a\rangle, |b\rangle, |c\rangle$ form a basis so any ket is a linear combination of them)

$$|1\rangle = \alpha_1|a\rangle + \beta_1|b\rangle + \delta_1|c\rangle$$
$$|2\rangle = \alpha_2|a\rangle + \beta_2|b\rangle + \delta_2|c\rangle$$
$$|3\rangle = \alpha_3|a\rangle + \beta_3|b\rangle + \gamma_3|c\rangle$$

we have:

$$\hat{H}|3\rangle = (E_0+2T)|3\rangle \Rightarrow \begin{pmatrix} -2T & T & T \\ T & -2T & T \\ T & T & -2T \end{pmatrix}\begin{pmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for normalization purposes

$$\begin{aligned}\hat{H}|1\rangle &= (E_0 - T)|1\rangle \\ \hat{H}|2\rangle &= (E_0 - T)|2\rangle\end{aligned} \Rightarrow \begin{pmatrix} T & T & T \\ T & T & T \\ T & T & T \end{pmatrix}\begin{pmatrix}\alpha_{1,2} \\ \beta_{1,2} \\ \delta_{1,2}\end{pmatrix} = 0 \qquad (i)$$

same equation
for $|1\rangle$ and $|2\rangle$ because
this eigenvalue ($E = E_0 - T$)
is degenerate

$$\Rightarrow \begin{pmatrix}\alpha_1 \\ \beta_1 \\ \delta_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -1 \\ 0\end{pmatrix}, \quad \begin{pmatrix}\alpha_2 \\ \beta_2 \\ \delta_2\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}1 \\ 1 \\ -2\end{pmatrix}$$

for normalization
purposes

There are many other solutions of eq. (i) above, any linear combination of the $|1\rangle, |2\rangle$ I found also works. I choose the first two I found (by trial and error) that were orthogonal to each other. Thus,

$$\boxed{|1\rangle = \frac{|a\rangle - |b\rangle}{\sqrt{2}}, \quad |2\rangle = \frac{|a\rangle + |b\rangle - 2|c\rangle}{\sqrt{6}}, \quad |3\rangle = \frac{|a\rangle + |b\rangle + |c\rangle}{\sqrt{3}}}$$

3. There are many ways of doing this. The simplest is, perhaps, write $|t=0\rangle$ as a combination of energy eigenstates (which evolve in time in a trivial way, simply by picking a phase $e^{-iEt/\hbar}$) This diagram may help visualize the strategy!

$$\begin{array}{ccc} |t=0\rangle & & |t\rangle \\ \| & & \| \\ c_1|1\rangle + c_2|2\rangle + c_3|3\rangle & \xrightarrow[\text{evolve in time}]{} & c_1 e^{-iE_1t/\hbar}|1\rangle + c_2 e^{-iE_2t/\hbar}|2\rangle + c_3 e^{-iE_3t/\hbar}|3\rangle \end{array}$$

$$|t=0\rangle = |a\rangle = \underbrace{\langle 1|a\rangle}_{c_1 = \frac{1}{\sqrt{2}}} |1\rangle + \underbrace{\langle 2|a\rangle}_{c_2 = \frac{1}{\sqrt{6}}} |2\rangle + \underbrace{\langle 3|a\rangle}_{c_3 = \frac{1}{\sqrt{3}}} |3\rangle$$

At a later time $t>0$:

$$|t\rangle = \frac{1}{\sqrt{2}} e^{-i(E_0-T)t/\hbar} |1\rangle + \frac{1}{\sqrt{6}} e^{-i(E_0-T)t/\hbar} |2\rangle + \frac{1}{\sqrt{3}} e^{-i(E_0+2T)t/\hbar} |3\rangle$$

The probability of measuring the "position" and finding "b" is given by

$$\text{prob. of } b = |\langle b|t\rangle|^2 = \left| \frac{1}{\sqrt{2}} e^{-i(E_0-T)t/\hbar} \langle b|1\rangle + \frac{1}{\sqrt{6}} e^{-i(E_0-T)t/\hbar} \langle b|2\rangle + \frac{1}{\sqrt{3}} e^{-i(E_0+2T)t/\hbar} \langle b|3\rangle \right|^2$$

$$= \left| -\frac{1}{2} e^{-i(E_0-T)t/\hbar} + \frac{1}{6} e^{-i(E_0-T)t/\hbar} + \frac{1}{3} e^{-i(E_0+2T)t/\hbar} \right|^2$$

$$= \frac{1}{9} \left| -e^{-i(E_0-T)t/\hbar} + e^{-i(E_0+2T)t/\hbar} \right|^2$$

$$= \frac{2}{9}\left(1 - \cos 3Tt/\hbar\right)$$

As a check, notice that the probability is 0 at $t=0$ (initial condition) and always smaller or equal to 1.

4. There are 3 measurements done:

1st measurement: "position" w/ result "b".
2nd measurement: energy w/ result $E_0 - T$
3rd measurement: "position"

state of the system after 1^{st} measurement: $|b\rangle$ (whatever the state was before the 1^{st} measurement, it collapsed to $|b\rangle$)

Probabilities for 2^{nd} measurement:

$E = E_0 + 2T$ w/ probability $= |\langle 3|b\rangle|^2 = 1/3$

energy eigenstate corresponding to this eigenvalue

state of the system at the time of 2^{nd} measurement

$$\boxed{E = E_0 - T \quad \text{w/ probability} = \underbrace{|\langle 1|b\rangle|^2 + |\langle 2|b\rangle|^2}_{\substack{\text{component of } |b\rangle \text{ on} \\ \text{the plane generated by } |1\rangle \text{ and } |2\rangle.}} = 1/3 + 1/3 = 2/3}$$

After the 2^{nd} measurement gives the result $E = E_0 - T$ the state collapses again:

$$|b\rangle \to \underbrace{\langle 1|b\rangle\, |1\rangle + \langle 2|b\rangle\, |2\rangle}_{\substack{\text{projection of } |b\rangle \text{ on the} \\ \text{plane generated by } |1\rangle \text{ and } |2\rangle}} = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{6}}|2\rangle$$

Normalizing this we get $-\frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2}|2\rangle$. The probability of measuring the "position" and finding "a" is given then by:

$$\text{probability of "a"} = \left|\langle a|\left(-\frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2}|2\rangle\right)\right|^2 = \left|-\sqrt{\frac{3}{8}} + \frac{1}{2\sqrt{6}}\right|^2 = \frac{1}{6}.$$

The measurement of the energy made the component along $|a\rangle$ to "regenerate" since it was not present initially.