Harmonic oscillator expectation values (P1) P401

HW#5 Fall 07

$$a \equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x + \frac{i}{m\omega}p\right), \quad a^\dagger \equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - \frac{ip}{m\omega}\right)$$

$$\Rightarrow \quad x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a^\dagger + a), \quad p = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(a^\dagger - a)$$

We want $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \langle p^2\rangle$ in n^{th} stationary state. So:

$$\langle n|x|n\rangle = \langle n|(a^\dagger + a)|n\rangle \cdot \left(\frac{\hbar}{2m\omega}\right)^{1/2}$$

Now using $a|n\rangle = \sqrt{n}\,|n-1\rangle$, and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, where $|n\rangle$ is the ket corresponding to the n^{th} state.

So $\langle n|x|n\rangle = \left\{\langle n|a^\dagger|n\rangle + \langle n|a|n\rangle\right\}\left(\frac{\hbar}{2m\omega}\right)^{1/2}$

$= \left\{\langle n|\sqrt{n+1}|n+1\rangle + \langle n|\sqrt{n}|n-1\rangle\right\}\left(\frac{\hbar}{2m\omega}\right)^{1/2}$, note $\langle n|m\rangle = \delta_{m,n}$,

$= 0$ since $\langle n|n+1\rangle = 0$, $\langle n|n-1\rangle = 0$.

So $\boxed{\langle n|x|n\rangle = 0}$

Similarly for $\langle p\rangle_n = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}\langle n|a^\dagger - a|n\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}\Big(\overbrace{\langle n|n+1\rangle}^{=0}\sqrt{n+1} + -1\cdot \underbrace{\langle n|n-1\rangle}_{=0}\sqrt{n}\Big)$

$\Rightarrow \boxed{\langle n|p|n\rangle = 0}$

Now $x^2 = \left[\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger)\right]^2 = \frac{\hbar}{2m\omega}\left(a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)$

We want to consider $\langle n|x^2|n\rangle$. So look at

$\langle n|a^2|n\rangle = \langle n|a\sqrt{n}|n-1\rangle = \sqrt{n}\sqrt{n-1}\langle n|n-2\rangle = 0$

similarly $\langle n|(a^\dagger)^2|n\rangle = 0$.

But $\langle n|aa^\dagger|n\rangle = \sqrt{n+1}\,\langle n|a|n+1\rangle = (n+1)\langle n|n\rangle = (n+1)$.

$\langle n|a^\dagger a|n\rangle = \sqrt{n}\,\langle n|a^\dagger|n-1\rangle$
$= \sqrt{n}\sqrt{n}\,\langle n|n\rangle = n.$

So putting it all together, $\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\left((n+1) + n\right)$

$$\boxed{\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}(2n+1)}$$

p2

~~$\langle p^2 \rangle$~~ Now $\langle n|p^2|n\rangle$ is very similar.

$$p^2 = -\frac{\hbar m\omega}{2}\left(a^2 + (a^+)^2 - aa^+ - a^+a\right)$$

So

$$\langle n|p^2|n\rangle = \frac{\hbar m\omega}{2}(2n+1)$$

So now $\sigma_p^2 = \frac{\hbar m\omega}{2}(2n+1)$, $\sigma_x^2 = \frac{\hbar}{2m\omega}(2n+1)$

and $\sigma_p^2\sigma_x^2 = \frac{\hbar^2}{4}(2n+1)^2$.

~~Clear~~ Clearly, this saturates $\sigma_p^2\sigma_x^2 \geq \hbar^2/4$ when $n=0$, that is, in the ground state.

Coherent States

① $[A, B^2] = AB^2 - B^2A = AB\cdot B - \underbrace{BA\cdot B + BA\cdot B}_{\text{add } 0} - B\cdot BA$

$= [A,B]B + B[A,B]$ qed.

② Show ~~that $[a_-, a_+^n]$~~ Show $[a, (a^+)^n] = n\, a_+^{n-1}$

~~$[a, (a^+)^{n-1}]$~~ Use a test function, $|m\rangle$.

$$[a, (a^+)^n]|m\rangle = a(a^+)^n|m\rangle - (a^+)^n a|m\rangle.$$

So $a\left((m+1)\cdots(m+n)\right)^{1/2}|m+n\rangle = a(a^+)^n|m\rangle$

$(m+n)\left((m+1)\cdots(m+n-1)\right)^{1/2}|m+n-1\rangle = a(a^+)^n|m\rangle$

and $(a^+)^n a|m\rangle = (a^+)^n\sqrt{m}\,|m-1\rangle = m\left((m+1)(m+2)\cdots(m+n-1)\right)^{1/2}|m+n-1\rangle$

So

$$[a,(a^+)^n]|m\rangle = n\sqrt{(m+1)\cdots(m+n-1)}\,|m+n-1\rangle$$

$$= n(a^+)^{n-1}|m\rangle$$

So $\boxed{[a,(a^+)^n] = n(a^+)^{n-1}}$, qed.

(3) Show that ~~XXX~~ $|\lambda\rangle = A e^{\lambda a_+}|0\rangle$,
where $|\lambda\rangle$ is a coherent state ($a_-|\lambda\rangle = \lambda|\lambda\rangle$).

Work on the right-hand side:

$$e^{\lambda a_+}|0\rangle = \left(1 + \lambda a_+ + \tfrac{1}{2}\lambda^2 a_+^2 + \cdots\right)|0\rangle$$

$$e^{\lambda a_+}|0\rangle = \left(|0\rangle + \lambda|1\rangle + \tfrac{1}{2}\lambda^2\sqrt{1\cdot 2}\,|2\rangle + \tfrac{1}{3!}\lambda^3\sqrt{3\cdot 2\cdot 1}\,|3\rangle + \cdots\right)$$

Now hit this with a_-. Then

$$a_- e^{\lambda a_+}|0\rangle = 0 + \lambda|0\rangle + \frac{\lambda^2}{2}\sqrt{2}\cdot\sqrt{2}\,|1\rangle + \frac{1}{3!}\lambda^3\sqrt{3\cdot 2\cdot 1}\cdot\sqrt{3}\,|2\rangle + \cdots$$

$$= \lambda\left(|0\rangle + \lambda|1\rangle + \frac{1}{2!}\lambda^2\sqrt{2\cdot 1}\,|2\rangle + \cdots\right)$$

$$= \lambda\left(1 + \lambda a^+ + \tfrac{1}{2!}\lambda^2 (a^+)^2\right)|0\rangle$$

$$= \lambda e^{\lambda a_+}|0\rangle$$

NOTE: Technically, this only shows $|\lambda\rangle$ is coherent, but not that all $|\lambda\rangle$ can be written like that $\Rightarrow$ see p5

$\Rightarrow e^{\lambda a_+}|0\rangle$ is a coherent state with eigenvalue λ.

I.e., $|\lambda\rangle = A e^{\lambda a_+}|0\rangle$, for some number A.

QED

(4) Show $A = e^{-|\lambda|^2/2}$ for a normalized $|\lambda\rangle$.

$$A^2\langle 0|\left(e^{\lambda a^+}\right)^+ e^{\lambda a^+}|0\rangle = 1 \qquad (\text{from } \langle\lambda|\lambda\rangle = 1).$$

$$\left(e^{\lambda a^+}\right)^+ = \left(1 + \lambda a^+ + \tfrac{\lambda^2}{2}(a^+)^2 + \cdots\right)^+ = 1 + \lambda^* a + \frac{(\lambda^*)^2}{2}a^2 + \cdots$$

$$= e^{\lambda^* a}$$

$A^2\langle 0| e^{\lambda^* a}|\lambda\rangle = 1$. But $e^{\lambda^* a}|\lambda\rangle = e^{\lambda^*\lambda}|\lambda\rangle$

$$= e^{\lambda^2}|\lambda\rangle$$

(4) continued

And finally, $A^2 \langle 0|\lambda\rangle \cdot e^{|\lambda|^2} = 1$

$$\Rightarrow A^2 e^{|\lambda|^2} \underbrace{\langle 0|}_{}(\underbrace{|0\rangle}_{1} + \underbrace{\lambda|1\rangle}_{0} + \dots) = 1$$

$$\Rightarrow \boxed{A = e^{-|\lambda|^2/2}}$$

Qed...

(5) Write $|\lambda\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$. Find c_n.

This is easy (if you use orthonormality of $|n\rangle$'s):

$$\langle n'|\lambda\rangle = \sum_{n=0}^{\infty} c_n \langle n'|n\rangle = \sum_{n=0}^{\infty} c_n \delta_{n,n'} = c_{n'}$$

But $\langle n|\lambda\rangle = A\langle n|\left[1 + \lambda a_+ + \frac{1}{2}\lambda^2 (a_+)^2 + \dots\right]|0\rangle$

$$= A \frac{1}{n!}\lambda^n \sqrt{n\cdot(n-1)(\dots)(1)}\underbrace{\langle n|n\rangle}_{1}$$

$$\boxed{c_n = \langle n|\lambda\rangle = e^{-|\lambda|^2/2} \frac{\lambda^n}{\sqrt{n!}}}$$

#3 continued

We've shown that $|\lambda\rangle = Ae^{\lambda a_+}|0\rangle$ is a coherent state.
Next, we can show that any coherent state can be written in that form.

$a_-|\lambda\rangle = \lambda|\lambda\rangle$ for any coherent state.

$|\lambda\rangle = \sum_n c_n |n\rangle$ for harmonic oscillator eigenstates $|n\rangle$, $c_n = \langle \lambda|n\rangle$.

Now $a_-|\lambda\rangle = \sum_n c_n a_-|n\rangle = \sum_n c_n \sqrt{n}\,|n-1\rangle = \sum_n c_n \sqrt{n} \frac{(a_+)^{n-1}}{\sqrt{(n-1)!}}|0\rangle$

$$= \lambda \sum_n c_n |n\rangle = \lambda \sum_n c_n \frac{(a_+)^n}{\sqrt{n!}}|0\rangle$$

so

$$\sum_n c_n \sqrt{n} \frac{(a_+)^{n-1}}{\sqrt{(n-1)!}}|0\rangle = \sum_n \lambda c_n \frac{(a_+)^n}{\sqrt{n!}}|0\rangle$$

$$\sum_n c_{n+1} \sqrt{n+1} \frac{(a_+)^n}{\sqrt{n!}}|0\rangle = \sum_n \lambda c_n \frac{(a_+)^n}{\sqrt{n!}}|0\rangle$$

$\longrightarrow$

So we have to solve the recurrence relation

$$\lambda c_n = c_{n+1}\sqrt{n+1}$$

This is easy: $c_n = \dfrac{\lambda^n}{\sqrt{n!}}\, c_0$, where c_0 is the "seed" (i.e., first #) in the series. (Can be proved by induction...).

So we see that

$$|\lambda\rangle = \sum c_0 \frac{\lambda^n}{n!}(a_+)^n|0\rangle = A e^{\lambda a_+}|0\rangle, \text{ where } A = c_0.$$

So we've shown that any $|\lambda\rangle$ which is a coherent state can be written as $|\lambda\rangle = A e^{\lambda a_+}|0\rangle$, and also that $A e^{\lambda a_+}|0\rangle$ is itself a coherent state. (I know, a bit redundant...).