Static Wave Packet

HW 4 Solution

(p1)

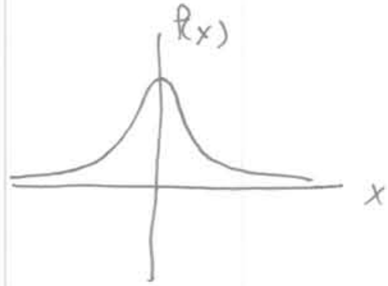
We're given $\psi(x) = Ae^{-\alpha x^2}$.

Normalize: $1 = \int dx\, \psi^*(x)\psi(x) = \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} A^2 \Rightarrow A^2 = \sqrt{\frac{2\alpha}{\pi}}$.

Find $\langle x \rangle$: $\langle x \rangle = \int dx\, \psi^*(x)\, x\, \psi(x) = \int_{-\infty}^{\infty} dx\, A^2 e^{-2\alpha x^2} x = 0$ by symmetry

Find $\langle p \rangle$: $\langle p \rangle = \int dx\, \psi^*(x)\left(-i\hbar \frac{\partial}{\partial x}\right)\psi(x) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}(-i\hbar \alpha x) e^{-\alpha x^2} = 0$ by symmetry.

Find $P(x) = |\psi(x)|^2 = \psi^*(x)\psi(x)$
$= A^2 e^{-2\alpha x^2} = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-2\alpha x^2}$

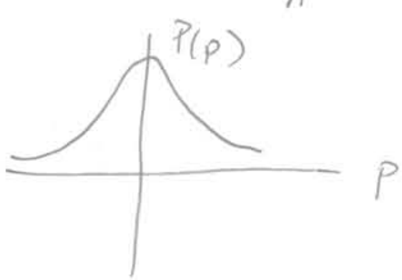


Find $P(p) = |\psi(p)|^2 = \psi^*(p)\psi(p)$

$\psi(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \psi(x)$

$= A\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} e^{-\alpha x^2} \underset{\text{complete square}}{=} \left(\frac{1}{2\pi\alpha}\right)^{1/4} \frac{1}{\hbar^{1/2}} e^{-p^2/4\alpha\hbar^2}$.

So $P(p) = \frac{1}{\hbar}\left(\frac{1}{2\pi\alpha}\right)^{1/2} e^{-p^2/2\alpha\hbar^2}$



Moving wave packet

(at some instant)

$\psi(x) = A e^{-\alpha x^2 - ikx}$

Normalize: $1 = A^2 \int_{-\infty}^{\infty} e^{-\alpha x^2 + ikx}\, e^{-\alpha x^2 - ikx}\, dx = \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2}$

$\Rightarrow A = \left(\frac{2\alpha}{\pi}\right)^{1/4}$. (as before).

Find $\langle x \rangle$: $\langle x \rangle = A^2 \int_{-\infty}^{\infty} e^{-\alpha x + ikx}\, x\, e^{-\alpha x - ikx}\, dx = A^2 \int_{-\infty}^{\infty} dx\, x\, e^{-2\alpha x^2} = 0$ by symmetry.

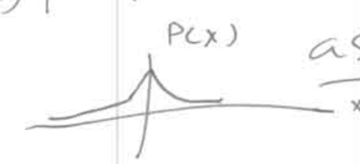
Find $\langle p \rangle$: $\langle p \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x)$

$= A^2 \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + ikx} \left(\underbrace{\cancel{+i\hbar \alpha x}}_{0 \text{ by symm}} + i^2 k \hbar\right) e^{-\alpha x^2 - ikx}$

$= -\hbar k A^2 \int dx\, e^{-2\alpha x^2} = -\hbar k = \langle p \rangle$.

Find $P(x) = |\psi(x)|^2 = \psi^*(x)\psi(x)$

$= \left(\frac{2\alpha}{\pi}\right)^{1/2} e^{-2\alpha x^2} \rightarrow$



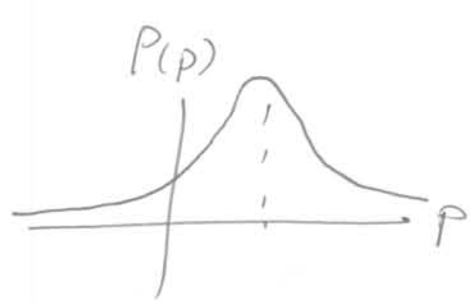
as before

Find $P(p) = \psi^*(p)\psi(p)$

So $\psi(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx} \psi(x) = \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty} dx\, e^{ipx} e^{-\alpha x^2 - ikx}$

$= \frac{1}{\hbar^{1/2}} \left(\frac{1}{2\pi\alpha}\right)^{1/4} e^{-(p/\hbar - k)^2/4\alpha}$

$\Rightarrow P(p) = \frac{1}{\hbar} \left(\frac{1}{2\pi\alpha}\right)^{1/2} e^{-(p/\hbar - k)^2/2\alpha} \Rightarrow$



Schrödinger Eq. In P-space

① Take $i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2}\psi(x,t)$

as schrödinger eq.

To go to momentum space, Fourier-transform both sides to momentum basis:

LHS:

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \underbrace{i\hbar\frac{\partial}{\partial t}}\psi(x,t) = i\hbar \frac{\partial}{\partial t} C(p,t)$$

$\longleftarrow$ pull out, FT has no t-dependence.

where $C(p,t) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \psi(x,t)$

RHS:

First, do schematically (drop constants), to see how it goes:

$$\int_{-\infty}^{\infty} dx\, e^{ipx} \frac{\partial^2}{\partial x^2}\psi(x) = \underbrace{e^{ipx}\frac{\partial}{\partial x}\psi(x)\Big|_{-\infty}^{\infty}}_{0} - ip\int_{-\infty}^{\infty} dx\, e^{ipx}\frac{\partial}{\partial x}\psi(x)$$

$$= \underbrace{-ip\, e^{ipx}\psi(x)\Big|_{-\infty}^{\infty}}_{0} + (ip)^2\int_{-\infty}^{\infty} dx\, e^{ipx}\psi(x)$$

So $\text{RHS} = (ip/\hbar)^2 \cdot \cancel{\left(\frac{1}{(2\pi\hbar)^{1/2}}\right)} \cdot \frac{-\hbar^2}{2M} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \psi(x,t)$

$= + p^2/2M\, C(p,t)$

So Schrödinger equation is now

$i\hbar \frac{\partial}{\partial t} C(p,t) = p^2/2M\, C(p,t)$, as you'd expect.

And $\frac{\partial}{\partial t} C(p,t) = \frac{-ip^2}{2M\hbar} C(p,t)$

$\Rightarrow C(p,t) = C(p,0)\, e^{-(ip^2/2M\hbar)t}$

$\longleftarrow$ initial condition at $t=0$.

Probability distribution of momentum

$P(p) = |C(p,t)|^2 = C^* C = |C(p,0)|^2$ (imaginary parts cancel)

So $P(p)$ doesn't change with time.

Finally, imagine adding a potential:

$$-i\hbar \frac{\partial}{\partial t} \psi(x,t) = \frac{-\hbar^2}{2M}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\,\psi(x,t)$$

Now RHS becomes $\frac{p^2}{2M} C(p,t) + \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \underbrace{V(x)\,\psi(x,t)}_{\text{convolution!}}$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} V(x) \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} C(p,t)$$

So

$$i\hbar \frac{\partial}{\partial t} C(p,t) = \frac{p^2 C(p,t)}{2M} + \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} V(x) \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} C(p,t)$$

and now this is a pain to solve!

(but we don't have to...)