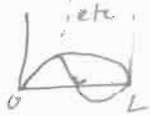
HW 3 Solutions | Fall 07, Aleksey Cherman — yell at me if anything's wrong.

(A) Find $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ for nth stationary state of ptcl in a box.

$\psi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$ is the wavefunction of the state



$$\langle x\rangle = \int_0^L dx \sqrt{\frac{2}{L}}\sin(n\pi x/L)\, x \,\sin(n\pi x/L)\cdot\sqrt{\frac{2}{L}}$$

$= L/2$ (use your favorite way of doing the integral - I used Mathematica...)

↑ ptcl lives @ $L/2$ on average

$$\langle x^2\rangle = \int_0^L dx\, \psi_n^*(x)\, x^2 \psi_n(x) = \frac{2}{L}\int_0^L dx \sin^2(n\pi x/L)\, x^2$$

$$= \frac{1}{6}L^2\left(2-\frac{3}{n^2\pi^2}\right) = L^2\left(\frac{1}{3}-\frac{1}{2n^2\pi^2}\right)$$

$$\langle p\rangle = \int_0^L dx\, \psi_n^*(x)\hat{p}\,\psi_n(x) = \frac{2}{L}\int_0^L dx \sin(n\pi x/L)\cdot i\hbar\frac{d}{dx}\sin(n\pi x/L)$$

$$= \frac{2i\hbar}{L}\int_0^L dx \sin(n\pi x/L)\, \frac{n\pi}{L}\cos(n\pi x/L) = 0$$ by orthogonality!

This makes sense - average velocity is 0.

$$\langle p^2\rangle = \int_0^L dx\, \psi_n^* \hat{p}^2 \psi_n(x) = \frac{2}{L}\cdot -\hbar^2 \int_0^L dx\, \psi_n(x)\cdot\frac{d^2}{dx^2}\psi_n(x)$$

$$= \frac{2}{L}\cdot +\hbar^2\int_0^L dx \sin^2(n\pi x/L)\cdot\left(\frac{n\pi}{L}\right)^2$$

$$\langle p^2\rangle = \frac{n^2\pi^2}{L^2}\hbar^2$$

Now can compute Δx^2, Δp^2:

$$\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = L^2\left(\frac{1}{3}-\frac{1}{2n^2\pi^2}\right) - \left(\frac{L}{2}\right)^2$$

$$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{n^2\pi^2\hbar^2}{L^2}$$

$\cancel{\text{So } \Delta x^2\Delta p^2 = \left(\frac{n^2\pi^2(2L-3)-3L}{6L}\right)\hbar^2 \text{, smallest with } n=1}$

(A) continued

Now $(\Delta x)^2 (\Delta p)^2 = \frac{1}{12}\hbar^2(n^2\pi^2 - 6) > \hbar^2/4$

This is smallest when $n=1$, and is clearly bigger than $\hbar^2/4$.

So ground state comes closest to saturating uncertainity relation.

(B) Ptcl in infinite square well has initial wavefunction

$$\psi(x,0) = A(\psi_1(x) + \psi_2(x))$$

① Normalize $\psi(x,0)$: $A^2 \int_0^L \psi(x,0)^2 dx = A^2 \int_0^L dx\, \underbrace{\psi_1(x)^2}_{1} + \underbrace{\psi_2(x)^2}_{1} + \underbrace{2\psi_1(x)\psi_2(x)}_{0 \text{ by orthogonality}} = 1$

$$\Rightarrow A^2(1+1) = 1 \Rightarrow \boxed{A = \sqrt{1/2}}$$

② Find $\psi(x,t)$, $|\psi(x,t)|^2$ use $\omega = \pi^2\hbar/2ma^2$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}} \sum_{n=1,2} e^{-in^2\omega t} \sin(n\pi x/L)$$

$$= \frac{1}{\sqrt{L}}\left(e^{-i\omega t}\sin(\pi x/L) + e^{-i4\omega t}\sin(2\pi x/L)\right).$$

$$|\psi(x,t)|^2 = \psi(x,t)^* \psi(x,t)$$

$$= \left(\frac{1}{\sqrt{L}}\right)^2\left[\sin(\pi x/L)^2 + \sin(2\pi x/L)^2 + 2\cos(3\omega t)\sin(\pi x/L)\sin(2\pi x/L)\right]$$

③ Now compute $\langle x \rangle$:

$$\int_0^L dx\, x\, |\psi(x,t)|^2 \xrightarrow[\text{algebra}]{} \frac{L}{18\pi^2}\left(9\pi^2 - 32\cos 3t\omega\right)$$

frequency $= 3\omega$ Amplitude $= L\cdot\frac{16}{9}$

$$\langle x^2 \rangle = \int_0^L dx\, x^2\, |\psi(x,t)|^2 = \frac{L^2}{144\pi^2}\left(48\pi^2 - 45 - 256\cos(t\omega)\right)$$

B continued

Suppose $\psi(x,0) = A(\psi_1(x) + e^{i\phi}\psi_2(x))$

Now $A^2 \int dx\, |\psi(x,0)|^2 = A^2(1+1) \Rightarrow A = \frac{1}{\sqrt{2}}$ as before.

Going through same steps as above,

$$\psi(x,t) = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}} \left(e^{-i\cdot 1\cdot \omega t} \sin(1\cdot\pi\cdot x/L) + e^{-i\cdot 4\cdot\omega t - i\phi} \sin(2\pi x/L) \right)$$

Now

$$\phi(x,t)^* \phi(x,t) = |\phi(x,t)|^2 = \sin(\pi x/L)^2 + \sin(2\pi x/L)^2 + 2\cos(3\omega t - \phi)\sin(\pi x/L)\sin(2\pi x/L)$$

And

$$\langle x \rangle = \int dx\, x\, |\psi(x,t)| = \frac{L}{18}\left(9 - \frac{32}{\pi^2}\cos(3\omega t - \phi)\right)$$

So Relative phases matter!

(C) $\psi(x,0) = A e^{-ax^2}$, free particle.

$A = \sqrt[4]{\frac{2a}{\pi}}$, as on previous HW.

Next, $\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k)\, e^{i(kx - \frac{\hbar k^2}{2m}t)}\, dk$

So we need $\psi(k) \Longrightarrow$ need Fourier Transform of $\psi(x)$.

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx}\, dx$$

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt[4]{\frac{2a}{\pi}} \int_{-\infty}^{\infty} e^{-(ax^2 + ikx)}\, dx$$, complete the square:

$a(x^2 + \frac{b}{a}x) = a\left[(x + \frac{1}{2}\frac{b}{a})^2 - (\frac{b}{2a})^2\right]$

$$= \frac{1}{\sqrt{2\pi}} \sqrt[4]{\frac{2a}{\pi}}\, e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} e^{-a(x + \frac{1}{2}b/a)^2}\, dx = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} e^{-k^2/a} \sqrt{\frac{\pi}{a}}$$

$\psi(k)$ ... $e^{-k^2/4a}$

(Problem C) continued

Next, we can get $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \, |\psi(x,t)|^2 \, dx = \left(\frac{2}{\pi}\right)^{1/2} w \cdot \frac{1}{4} w^{-3} \sqrt{\pi/2}$$

$$\boxed{\langle x^2 \rangle = \frac{1}{4}\frac{1}{w^2}}$$

We want to compute $\langle p^2 \rangle$. First, let's go back to

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{(1-i\alpha)^{1/2}}{(1+\alpha^2)^{1/2}} \exp\left[-\underbrace{\frac{a(1-i\alpha)}{1+\alpha^2}}\, x^2\right], \quad \alpha \equiv \frac{2\hbar a t}{m}$$

$$\beta \equiv \frac{a(1-i\alpha)}{1+\alpha^2}$$

Now $\frac{d^2}{dx^2}\psi = -2\beta\,\psi(x,t) + 4\beta^2 x^2 \psi(x,t)$

So $$\langle p^2 \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x,t)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\psi(x,t)$$

$$= -\hbar^2 \int_{-\infty}^{\infty} dx\,(-2\beta)\,|\psi(x,t)|^2 + -\hbar^2 \int_{-\infty}^{\infty} dx \; x^2 (4\beta^2)\, |\psi(x,t)|^2$$

$$= -\hbar^2 \left(\frac{2}{\pi}\right)^{1/2} w \cdot (-2\beta)\sqrt{\frac{\pi}{2w^2}} \quad -\hbar^2 (4\beta^2)\left(\frac{1}{4}\frac{1}{w^2}\right)$$

$$\langle p^2 \rangle = \hbar^2\left(\frac{-\beta^2}{w^2} + 2\beta\right) = \hbar^2\left(\frac{a}{1+\alpha^2} + \frac{a\alpha^2}{1+\alpha^2}\right) = \boxed{\hbar^2 a = \langle p^2 \rangle}$$

(Trying to do all this without defining away intermediate constants as I did above is hard (at least for me) – the algebra gets seriously nasty!)

Now $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{4w^2}$, $\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle$

$\sigma_p^2 = \hbar^2 a$

So $\sigma_x^2 \sigma_p^2 = \frac{\hbar^2 a}{4w^2}$

$$\boxed{\sigma_x^2 \sigma_p^2 = \frac{\hbar^2}{4}\left(1 + (2\hbar a t/m)^2\right)}, \text{ smallest @ } t=0$$

at $t=0$ $\sigma_x^2 \sigma_p^2 = \hbar^2/4$, hits uncertainty limit. Then wavefunction spreads...

$\boxed{\text{Part C}}$ continued

$$\phi(k) = e^{-k^2/4a} \cdot \frac{1}{(2\pi a)^{1/4}} \qquad \psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk$$

Algebra's a bit hairy here, but it's doable.

$$\psi(x,t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} dk \exp\left(-k^2/2a + i\left(kx - \frac{\hbar k^2}{2m} t\right)\right)$$

Now take $\underbrace{-\left(\frac{1}{2a} + \frac{i\hbar t}{2m}\right)}_{\alpha} k^2 + (\underbrace{ix}_{\beta}) k = -\alpha(k^2 - \beta/\alpha\, k) = -\alpha\left\{\left(k - \frac{\beta}{2\alpha}\right)^2 - \left(\frac{\beta}{2\alpha}\right)^2\right\}$

$$\psi(x,t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{(2\pi a)^{1/4}} \cdot e^{\beta^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(k - \beta/2\alpha)^2} dk$$

(b) $= \frac{1}{(2\pi)^{1/2}} \frac{1}{(2\pi a)^{1/4}} e^{+\beta^2/4\alpha} \sqrt{\frac{\pi}{\alpha}}$, now plug α, β back in, simplify...

$$\boxed{\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \left(1 + \frac{2i\hbar a t}{m}\right)^{-1/2} \exp\left[\frac{-a x^2}{1 + 2i\hbar a t/m}\right]}$$

(c) Now we want $|\psi|^2$. Do some algebra:

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} (1 + i\alpha)^{-1/2} \exp\left[-\frac{a x^2}{1 + i\alpha}\right], \quad \alpha \equiv \frac{2\hbar a t}{m}$$

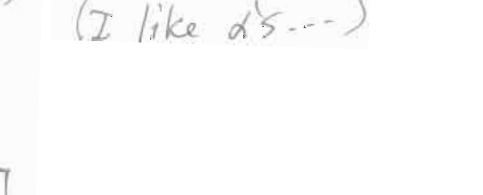
(I like α's...)

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{(1 - i\alpha)^{1/2}}{(1 + \alpha^2)^{1/2}} \exp\left[\frac{-a(1 - i\alpha)x^2}{1 + \alpha^2}\right]$$

So $\psi^*(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{(1 + i\alpha)^{1/2}}{(1 + \alpha^2)^{1/2}} \exp\left[-\frac{a(1 + i\alpha)x^2}{1 + \alpha^2}\right]$

and $\psi^*\psi = |\psi|^2 = \left(\frac{2a}{\pi}\right)^{1/2} \frac{(1 + \alpha^2)^{1/2}}{1 + \alpha^2} \exp\left[-\frac{2a x^2}{1 + \alpha^2}\right]$, let $w^2 \equiv \frac{a}{1 + \alpha^2}$,

$$\boxed{|\psi(x,t)|^2 = \left(\frac{2}{\pi}\right)^{1/2} w \exp\left[-2w^2 x^2\right]}$$



Now it's immediately clear that $\boxed{\langle x \rangle = \langle p \rangle = 0}$ by symmetry!

$$\left(\text{since } \int_{-\infty}^{\infty} x e^{-x^2} = 0\right)$$

(D)

(a) $\int_{-\infty}^{\infty} dx\, (x^3-1)\, \delta(x-1) = 1^3 - 1 = 0$

(b) $\int_{-\infty}^{\infty} dx\, \delta(cx) = \frac{1}{c}\int dx\, \delta(x) = \frac{1}{c} \cdot 1 = 1/c$

$x = cx'$
$dx = c\,dx'$

So $\int dx\, \delta(cx) = \int dx\, \frac{1}{c}\, \delta(x) \Rightarrow \delta(cx) = \frac{1}{c}\delta(x)$.

(c) To show $\frac{d\Theta(x)}{dx} = \delta(x)$, use Fundamental Thm of calculus:

$f(x) = \int_{-a}^{\cancel{x}} dx\, \delta(x) = 0$, for any $a>0$, and $\cancel{x}>0$.

while $f(x) = \int_{-\cancel{\infty}a}^{x} \delta(a) = 1$ for any $-a>0$, $x>0$

So $f(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$. By fund thm, $\frac{df(x)}{dx} = \frac{d\Theta(x)}{dx} = \delta(x)$.

(d) $F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx}\, \delta(x) = \frac{e^{-ik\cdot 0}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$

And now $\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{ikx} \cdot F(k)$

$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}$

$\int u\,dv = uv - \int v\,du$

(e) $\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x) = f(x)\,\delta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} \delta(x)\, f'(x)\, dx$

$= 0 - f'(0)$

$\Rightarrow \int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x) = -f'(0)$.

(Part D continued)

(f) Let $\delta(x) \equiv \lim_{\alpha \to \infty} \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}$.

Need to show that $\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{+ikx}$

$$F(k) = \lim_{\alpha \to \infty} \int_{-\infty}^{\infty} \frac{d\cancel{x}}{(2\pi)^{1/2}}\, e^{-ikx} \cdot \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}$$ complete square again...

$$= \lim_{\alpha \to \infty} \sqrt{\frac{\pi}{2\alpha}}\, e^{-k^2/4\alpha}$$

And now $\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^{1/2}}\, e^{+ikx} \sqrt{\frac{\pi}{2\alpha}}\, e^{-k^2/4\alpha} = \lim_{\alpha \to \infty} \sqrt{\frac{\pi}{\alpha}}\, e^{-\alpha x^2}$

$= \delta(x)$. QED.

- ~~HW:~~ People aren't using orthogonality to do problems
- Deep confusion on $\delta(x)$ and relation to $\Theta(x)$.
  → also on what's meant by a proof.
- People for some reason thought that $\langle p^2 \rangle = a\hbar^2$ was a given on free ptcl problem, not something to compute.
- $e^{i\omega t} + e^{-i\omega t} = 2\cos(\omega t)$
- $|\psi(x,t)|^2$ can't be complex, ever..
- people still don't understand the notion of a proof: To prove $A = B$, manipulate A by itself until you get B.
- Lots of people not showing their steps, I think copying is possible... –
- $|\psi(x,t)|^2 \neq \int_{-\infty}^{\infty} dx \; \psi^* \psi$!