# Semi-classical quantization for polynomial $V$

HW2, Phys 401
Solution by AC, P1

Recall that $\boxed{L = j\hbar}$, where $j \in \mathbb{Z}$ and $L$ is angular momentum.
Also, classically, $L = mvr$, $m$: mass, $v$: velocity, $r$: radius

Suppose we've a potential $V(r) = \dfrac{\alpha}{r^n}$.
Then $F = ma = -\dfrac{\partial V(r)}{\partial r} = \boxed{\dfrac{\alpha n}{r^{n+1}} = ma}$

Next, what are the allowed radii?
$F = mv^2/r$ for circular motion (assuming orbits are circular)
So using above relations and doing some algebra,

$$\frac{|\alpha| n}{r^{n+1}} = mvr \cdot \frac{v/r}{r} = L\frac{v}{r^2} = \frac{L^2}{mr^3}, \text{ since } v = \frac{L}{mr}.$$

Note: use $|\alpha|$ here and not $\alpha$ (Get vector directions right!)

Using ang. momentum quantization, we find

$$\frac{|\alpha| n}{r^{n+1}} = \frac{(j\hbar)^2}{mr^3}$$

Solving for $r$, we obtain

$$r^{2-n} = \frac{(j\hbar)^2}{mn|\alpha|} \Longrightarrow \boxed{r(j) = \left[\frac{(j\hbar)^2}{mn|\alpha|}\right]^{\frac{1}{2-n}}}$$

Next, what are the allowed energies?

$E = \frac{1}{2}mv^2 + V(r)$ classically

$\frac{1}{2}mv^2 = \dfrac{L^2}{2mr^2}$, and now we can solve for $E(j)$.

$$\boxed{E(j) = \frac{(j\hbar)^2}{2m\, r(j)^2} + \frac{\alpha}{r(j)^n}}$$, where $r(j)$ is given above.

Note: For $n=1$, $\alpha = -ke^2$, this reduces to

$$\boxed{E(j) = -\frac{1}{j^2}\frac{(ke^2)^2 m}{2\hbar^2}}$$ (Bohr model)

# A look @ the uncertainity principle

HW2, Phys 401
Solution p 2

$\psi(x) = A e^{-\alpha x^2}$ for some particle.

Normalize ( find $A$ s.t. $\int_{-\infty}^{\infty} dx\, \psi(x)^* \psi(x) = 1$ )

So $A^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} = 1 \Rightarrow A^2 \cdot \sqrt{\frac{\pi}{2\alpha}} = 1 \Rightarrow \boxed{A = \left(\frac{2\alpha}{\pi}\right)^{1/4}}$

Next, need to compute $\langle x \rangle$, $\langle x^2 \rangle$, $\sigma_x^2 \equiv \langle (x - \langle x \rangle)^2 \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \psi(x) = A^2 \int_{-\infty}^{\infty} dx\, x\, e^{-2\alpha x^2}$$

But $x$ is ~~even~~ odd function, $e^{-2\alpha x^2}$ is even and integral goes from $(-\infty, \infty)$. So $(-\infty, 0)$ part cancels $(0, \infty)$ part

$\Rightarrow \boxed{\langle x \rangle = 0}$.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x^2 \psi(x) = A^2 \int_{-\infty}^{\infty} dx\, x^2 e^{-2\alpha x^2}$$

Trick (general!): $x^2 e^{-2\alpha x^2} = -\frac{1}{2} \frac{d}{d\alpha}\left(e^{-2\alpha x^2}\right)$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, x^2 e^{-2\alpha x^2} = -\frac{1}{2} \frac{d}{d\alpha} \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} = -\frac{1}{2} \frac{d}{d\alpha} \left(\sqrt{\frac{\pi}{2\alpha}}\right) = \frac{1}{2} \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \alpha^{-3/2}$$

So $\boxed{\langle x^2 \rangle = \frac{1}{4\alpha}}$ , since $A^2 = \sqrt{\frac{2\alpha}{\pi}}$

And finally $\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 + \langle x \rangle^2 - 2\langle x \rangle x \rangle$

$= \langle x^2 \rangle + \langle x^2 \rangle - 2 \langle x \rangle \langle x \rangle = \langle x^2 \rangle - \langle x \rangle^2$

$\Rightarrow \boxed{\sigma_x^2 = \frac{1}{4\alpha}}$

Now do <u>momentum</u>: calculate $\langle p\rangle$, $\langle p^2\rangle$, $\sigma_p^2$.

$$\langle p\rangle \equiv \int dx\, \psi^*(x)\left(-i\hbar \frac{d}{dx}\right)\psi(x) = A^2\int dx\, e^{-\alpha x^2}\left(-i\hbar(-2\alpha x)\right)e^{-\alpha x^2}$$

$$= A^2 i\hbar(2\alpha)\underbrace{\int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2}\, x}_{\text{done before}} = 0$$

$$\langle p^2\rangle = \int dx\, \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)^2\psi(x) = A^2\cdot -i^2\hbar^2\cdot(2\alpha)^2\underbrace{\int_{-\infty}^{\infty} dx\, x^2 e^{-2\alpha x^2}}_{\text{done before}}$$

$$= +\underbrace{A^2}_{=\sqrt{2\alpha/\pi}}\hbar^2(2\alpha^{\cancel{2}})^2\cdot\left(\frac{1}{4}\sqrt{\frac{\pi}{2}}\,\alpha^{-3/2}\right)$$

algebra

$$\boxed{\langle p^2\rangle = +\hbar^2\alpha}$$

So $\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \boxed{+\hbar^2\alpha = \sigma_p^2}$

Recall that $\sigma_x^2 = \dfrac{1}{4\alpha}$.

So as $\alpha\to\infty$, $\sigma_x\to 0$, $\sigma_p\to\infty$
as $\alpha\to 0$ $\sigma_p\to 0$, $\sigma_x\to\infty$

Note that

$$\boxed{\sigma_x\sigma_p = \hbar/2}$$

# Ehrenfest's Theorem

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Solution p 4

$$\frac{\partial}{\partial t}\langle p\rangle = \frac{\partial}{\partial t}\int dx\, \psi^*(x)\,\hat p\,\psi(x)$$

$$= \int dx \left(\frac{\partial}{\partial t}\psi^*(x)\right)\hat p\,\psi(x) + \int dx\, \psi^*(x)\,\hat p\left(\frac{\partial}{\partial t}\psi(x)\right)$$

$$+ \cancel{\int dx\, \psi^*(x)\left(\underbrace{\frac{\partial}{\partial t}\hat p}_{0 \text{ in Schroedinger picture}}\right)\psi(x)}$$

Using $\hat H\psi(x) = i\hbar\frac{\partial}{\partial t}\psi(x)$ (assuming $\psi$ has $t$-dependence)

$$\frac{\partial\langle p\rangle}{\partial t} = \frac{1}{i\hbar}\int dx\, \psi^*(x)\,\hat p\,\hat H\,\psi(x) - \frac{1}{i\hbar}\int dx\,\psi^*(x)\,\hat H\,\hat p\,\psi(x)$$

since $\psi^*(x)\hat H = -i\hbar\frac{\partial}{\partial t}\psi^*(x)$ ($\hat H$ is Hermitian $\Rightarrow \hat H^* = \hat H$.)

So $\frac{\partial\langle p\rangle}{\partial t} = \frac{1}{i\hbar}\int dx\,\psi^*[\hat p,\hat H]\psi(x)$. But $[\hat p,\hat H] = [\hat p, \hat p^2/2m + \hat V(x)]$
$= [\hat p, \hat V(x)]$

giving $\frac{\partial\langle p\rangle}{\partial t} = \frac{1}{\cancel{i\hbar}}\left\{\int dx\,\psi^*\left(-\cancel{i\hbar}\frac{\partial}{\partial x}\right)[V(x)\psi(x)] - \int dx\,\psi^*(x)\,V(x)\frac{\partial}{\partial x}\cdot(-\cancel{i\hbar})\,\psi(x)\right\}$

$$= \int_{-\infty}^{\infty} dx\,\psi^*(x)V(x)\frac{\partial}{\partial x}\psi(x) - \int_{-\infty}^{\infty} dx\,\psi^*(x)\frac{\partial}{\partial x}[V(x)\psi(x)]$$

$$= \cancel{\int_{-\infty}^{\infty} dx\,\psi^*(x)\,V(x)\frac{\partial}{\partial x}\psi(x)} - \int_{-\infty}^{\infty} dx\,\psi^*(x)\left[\frac{\partial}{\partial x}V(x)\right]\psi(x) - \cancel{\int_{-\infty}^{\infty} dx\,\psi^*(x)\,V(x)\frac{\partial}{\partial x}\psi(x)}$$

So
$$\boxed{\frac{\partial\langle p\rangle}{\partial t} = \int_{-\infty}^{\infty} dx\,\psi^*(x)\left[-\frac{\partial}{\partial x}V(x)\right]\psi(x) = \left\langle -\frac{\partial V(x)}{\partial x}\right\rangle}$$

This is precisely Newton's Law, $\frac{dp}{dt} = -\frac{\partial V}{\partial x}$
in terms of expectation values.