QUANTUM PHYSICS I

PROBLEM SET 1

due ...

A. Dimensional analysis and the blackbody radiation

i) Writting

$$[T^{\alpha}\nu^{\beta}c^{\gamma}k^{\delta}] = [\frac{temperature^{\alpha-\delta}mass^{\delta}length^{2\delta+\gamma}}{time^{2\delta+\beta+\gamma}}] = [1] \tag{1}$$

we find that $\alpha = \beta = \gamma = \delta = 0$.

ii) The equation

$$[\rho_T(\nu)] = [\frac{mass}{length\ time}] = [T^{\alpha}\nu^{\beta}c^{\gamma}k^{\delta}] = [\frac{temperature^{\alpha-\delta}mass^{\delta}length^{2\delta+\gamma}}{time^{2\delta+\beta+\gamma}}] \tag{2}$$

implies $\delta = 1, \alpha = 1, \gamma = -3$ and $\beta = 2$. Thus

$$\rho_T(\nu) \sim \frac{kT\nu^2}{c^3} \tag{3}$$

which, up to the overall normalization, is the Rayleigh-Jeans result.

iii) A dimensionless combination is $h\nu/kT$. Indeed

$$[\frac{h\nu}{kT}] = [\frac{energy\ time}{time}\frac{temperature}{energy\ temperature}] = [1]. \tag{4}$$

iv)

$$[\frac{h}{E}] = [\frac{energy\ time}{energy}] = [time]. \tag{5}$$

B. Stefan-Boltzmann law

i) We use the change of variables $x = h\nu/kT$ to arrive at

$$R_T = \frac{c}{4}\rho_T(\nu) = \int_0^{\infty}\frac{2\pi h\nu^2}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1} \tag{6}$$

$$= \frac{2\pi(kT)^4}{c^2h^3}\underbrace{\int_0^{\infty}\frac{x^3}{e^x-1}}_{T\ independent\ constant} \tag{7}$$

ii) Using that

$$\int_0^{\infty}\frac{x^3}{e^x-1} = \frac{\pi^4}{15} \tag{8}$$

and the result in i) we find

$$R_T = \frac{2\pi^5k^4}{15c^2h^3}T^4 \tag{9}$$

and $\sigma = 2\pi^5/(15c^2h^3)$.

C. Blackbody radiation and the temperature of the Earth

i) The total power radiated by the sun is the product of the power radiated per unit area times the total surface area of the sun:

$$\textit{total power} = 4\pi R_s^2 \sigma T_s^4 \tag{10}$$

ii) The light emitted by the sun spreads in all directions and only a small fraction reaches the Earth. When the light reaches a distance r from the sun, it is spread over an area of $4\pi r^2$. The Earth occupies only an area of πR_E^2 (its cross section) of this surface. In addition, a fraction of a of the light shinning on the Earth surface is reflected and $1 - a$ is absorbed. We find then

$$\textit{power absorbed} = 4\pi R_s^2 \sigma T_s^4 \frac{\pi R_E^2}{4\pi r^2}(1-a). \tag{11}$$

iii) In equilibrium

$$4\pi R_s^2 \sigma T_s^4 \frac{\pi R_E^2}{4\pi r^2}(1-a) = 4\pi R_E^2 \sigma T_E^4. \tag{12}$$

Solving for T_E we find

$$T_E = (1-a)^{1/4}\sqrt{\frac{R_s}{2r}}T_s \approx (1-0.3)^{1/4}\sqrt{\frac{7.10^8 m}{300.10^9 m}}5800K \approx 251\ K \approx -22\ C. \tag{13}$$

This calculation would seem more impressive if we input the Earth's average temperature $15C$ and compute the sun's temperature. The result comes out right to within a few percent.