QUANTUM PHYSICS I

PROBLEM SET 3

due October 10, 2007

A. (Griffiths, 2.4) Uncertainty on the square well states

Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$ and $\langle p^2 \rangle$ for the nth stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closer to the uncertainty limit ?

B. (Griffiths, 2.5 and 2.6, sort of ...) When is the wave function phase relevant ?

A particle on an infinite square well has an inital wave function that is an equal superposition of the two first states:

$$\Psi(x,0) = A(\psi_1(x) + \psi_2(x)). \tag{1}$$

i) Normalize $\Psi(x,0)$.
ii) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. To simplify the result, let $\omega = \pi^2\hbar/(2ma^2)$.
iii) Compute $\langle x \rangle$ and notice it is oscillatory. What is the amplitude and angular frequency of this oscillation ?
Although the overall phase of the wave function is of no physical significance (it cancels out whenever you calculate a measurable quantity), the relative phase of the coefficients in eq. (1) does matter. For example, suppose we take

$$\Psi(x,0) = A(\psi_1(x) + e^{i\psi}\psi_2(x)) \tag{2}$$

instead of eq. (1). Find
iv) $\Psi(x,t)$,
v) $|\Psi(x,t)|^2$ and
vi) $\langle x \rangle$
and compare with the $\phi = 0$ case.

C. (Griffths 2.22) The gaussian wave packet

A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-ax^2}, \tag{3}$$

where A and a are constants (and a is real and positive).
a) Normalize $\Psi(x,0)$.
b) Find $\Psi(x,t)$ Griffths gives away the answer so I might as well do the same:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+(2i\hbar at/m))}}{\sqrt{1+(2i\hbar at/m}}. \tag{4}$$

c) Find $|\Psi(x,t)|^2$ and express your answer in terms of

$$w = \sqrt{\frac{a}{1+(2\hbar at/m)^2}}. \tag{5}$$

Sketch $|\Psi|^2$ as a function of x at $t=0$ andfor some large t. Qualitatively, what happens to $|\Psi|^2$ as time goes on ?
d) Find $\langle x \rangle, \langle p \rangle, \langle x^2 \rangle, \langle p^2 \rangle, \sigma_x$ and σ_p. Partial answer: $\langle p^2 \rangle = a\hbar^2$.
e) Does the uncertainty principle hold ? At what time does the the system come closest to the uncertainty limit ? Do you have a physical understanding of why the uncertainty evolves in time the way it does ?

D. All you wanted to know about Dirac's δ-function and were afraid to ask

We can define the δ-function buy its behavior inside integrals:

$$\int_a^b f(x)\delta(x-y) \equiv f(y), \quad \text{for} \quad a < y < b, \tag{6}$$

for any well-behaved function $f(x)$, usually called the *test-function.* Show that
a) $\int_{-\infty}^{\infty} dx(x^3-1)\delta(x-1) = 0$
b) $\delta(cx) = \frac{1}{|c|}\delta(x)$ (Hint: Insert both sides of the equation in the definition of δ above and change variables.)
c) $\frac{d\theta(x)}{dx} = \delta(x)$ where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x < 0. \end{cases} \tag{7}$$

(and $\theta(0) = 1/2$ if it ever matters).
d) What is the Fourier transform of $\delta(x)$

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\delta(x) = ? \tag{8}$$

Use Plancherel's theorem (see text) to show that

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}, \tag{9}$$

which is a relation we used in class after a hand waving "proof".
e) Show that

$$\int_{-\infty}^{\infty} dx f(x)\delta'(x) = -f'(0). \tag{10}$$

Feel free to assume that $f(x) \to$ as fast as necessary as $x \to \pm\infty$.
f) Another way of defining the δ-function is through the relation

$$\delta(x) \equiv \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}. \tag{11}$$

Show that the result of d) is the same using this new definition. Feel free to exchange the order of limits and integrations and assume that the test functions are as well behaved as necessary.