QUANTUM PHYSICS I
PROBLEM SET 1
due September 15, 2006

A. Dimensional analysis and the blackbody radiation

Using simple dimensional analysis we can derive many results of the (flawed) classical theory of blackbody radiation. In particular we can deduce it is bounded to fail ! Follow the steps below.

i) As we showed in class the energy density inside the blackbody cavity $\rho_T(\nu)$ is a function only of the temperature and the frequency. In addition it can depend on the fundamental constants c (speed of light, appearing on the Maxwell equations) and k (Boltzman constant). Convince yourself that the dimensions of these quantities are $[T] = temperature$, $[\nu] = 1/time$, $[c] = length/time$ and $[k] = energy/temperature = mass\ length^2/time^2\ temperature$ ([..] denotes the dimension of ...) . Then prove that no dimensionless quantity can be created using these quantities (hint: write $T^\alpha \nu^\beta c^\gamma k^\delta$ and prove that no choice of $\alpha, ...\delta$ can produce a dimensionless quantity). This result shows that non-polynomial functions like $\exp(x) = 1 + x + x^2/2 + \cdots$ cannot arise since there is no dimensionless x to serve as an argument for exp.

ii) Use the same kind of argument show that $\rho_T(\nu)$, which has dimensions of $energy/volume\ frequency = mass/length\ time$, is proportional to ν^2. This result was "derived" in class through a more complicated calculation. Show that the total energy, integrated over all frequencies, diverges.

iii) The introduction of another fundamental constant h with dimensions of *energy time* invalidate the argument in i). Find a dimensionless combination of h, ν, k and T. This quantity can appear inside non-polynomial functions and invalidate the result in ii).

iv) The Planck constant can be combined with momentum p to form a length: h/p. Show how to combine h and an energy to form a quantity with units of time.

B. Stefan-Boltzmann law

Special note: I believe I had a numerical factor wrong in class. The relation between the energy radiated by a blackbody per unit area per unit time per unit frequency ($R_T(\nu)$) and the energy density per unit frequency inside the cavity ($\rho_T(\nu)$) is $R_T(\nu) = c\rho(\nu)/4 = 2\pi h\nu^3/c^2\ 1/(e^{h\nu/kT} - 1)$.

i) Without computing any integral, show that the total energy emitted by a blackbody (per unit area and unit time) is proportional to T^4. This is known as Stefan's law.

ii) Integrate Planck's formula over frequencies to find that the total energy emitted by a blackbody is $R_T = \sigma T^4$. Determine σ.

C. Blackbody radiation and the temperature of the Earth

In this problem we will make a simple model of the global energy balance of the Earth and estimate its temperature. The main source of energy on Earth is the sun. Both the sun and the Earth can be approximated by blackbodies, but the Earth reflects, instead of absorbing, about 35% of the energy received (it is said the *albedo* of the Earth is 0.3).

i)Calculate the power (energy per unit time) the sun radiates in terms of its surface temperature T_s and radius R_s.

ii) Find the fraction of the energy above absorbed by the Earth as a function of its albedo a and distance from the sun r and the Earth radius R_E. (hint: the energy emitted by the sun is spread over a sphere of radius r when it reaches the Earth and only a fraction shines upon the Earth).

iii) In equilibrium, the amount of energy absorbed by the Earth is emitted back to space. Equate the result from ii) to the power emitted by the Earth ($4\pi R_E^2 \sigma T_E^4$) and find the temperature of the Earth in equilibrium. Plug the values: $r = 150 \times 10^9 m$
$R_s = 7 \times 10^8 m$
$T_s = 5.800K$
to find the temperature of the Earth at equilibrium. The temperature you'll find is reasonably close to the correct average temperature of the Earth, but a little lower. The main ingredient missing in our model is the fact that some gases (mostly carbon dioxide) in the Earth's atmosphere absorb some of the radiation emitted by the Earth, acting as a blanket and making it warmer. This is called the "greenhouse effect". The temperature of the Earth depends crucially on the amount of these gases in the atmosphere.

D. The cosmic microwave background (CMB) and the Sunyaev-Zeldovich effect

We mention in class that the Universe is filled with blackbody radiation of 2.7 K left over from the Big-Bang. High energy electrons can sometimes kick the photons in the CMB to higher energies and deform the blackbody spectrum. This is a kind of inverse Compton scattering and the resulting deformation of the CMB spectrum can be used to detect the existence of clusters of galaxies at extremely large distances (Sunyaev-Zeldovich effect). Calculate the frequency shift for a typical photon with energy kT caused by a collision with a *non-relativistic* electron. Hint: Verify that the typical photon in the CMB has an energy much smaller than the rest mass of the electron $mc^2 \approx 0.55 \times 10^6$ eV and then disregard terms of order of the photon energy compared to the electron rest mass and the photon momenta compared to the electron momentum. *You can assume that the photon is back-scattered ($\phi = \pi$) and the collision is back-to-back ($\theta = \pi$)).*