PROBLEM II

1)
$$\left[\frac{\hat{p}_r^2}{2M} + \frac{\hat{L}^2}{2Mr^2} + \underbrace{V(r)}_{\text{no } \theta,\phi \text{ dependence}}\right] \frac{u(r)}{r} Y_\ell^m(\theta,\phi) = E \frac{u(r)}{r} Y_\ell^m(\theta,\phi)$$

$$\Downarrow$$

$$Y_\ell^m \frac{\hat{p}_r^2}{2M}\frac{u(r)}{r} + \frac{u(r)}{r}\frac{\ell(\ell+1)\hbar^2}{2Mr^2}Y_\ell^m + V(r)\frac{u(r)}{r}Y_\ell^m(\theta,\phi) = E\frac{u(r)}{r}Y_\ell^m(\theta,\phi)$$

But
$$\hat{p}_r^2 \frac{u(r)}{r} = -\hbar^2 \frac{1}{r}\frac{d}{dr}\cancel{r}\frac{1}{\cancel{r}}\frac{d}{dr}\cancel{r}\frac{u(r)}{\cancel{r}}$$
$$= -\hbar^2 \frac{1}{r}\frac{d^2}{dr^2}u(r)$$

so
$$\boxed{-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}u(r) + \left(\frac{\ell(\ell+1)\hbar^2}{2Mr^2} + V(r)\right)u(r) = E\,u(r)}$$

2) For $\ell=0$:

$$-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}u(r) + V(r)\,u(r) = E\,u(r)$$

$a<r<b$: $\quad -\frac{\hbar^2}{2M}\frac{d^2}{dr^2}u(r) = E\,u(r) \Rightarrow u(r) = A\cos kr + B\sin kr$,
with $E = \hbar^2k^2/2M$

boundary conditions: $u(r=a)=0 \Rightarrow A\cos ka + B\sin ka = 0 \quad$ (i)
$u(r=b)=0 \Rightarrow A\cos kb + B\sin kb = 0 \quad$ (ii)

Eliminate B from (ii) and plug back to (i):

$$0 = A\cos ka - A\frac{\cos kb}{\sin kb}\sin ka \Rightarrow \cos ka\sin kb - \cos kb\sin ka = +\sin k(b-a) = 0$$