SEPARATION OF VARIABLES IN SPHERICAL COORDINATES

Schrödinger equation

$$\left[-\frac{\hbar^2}{2M}\nabla^2 + V\right]\varphi = E\varphi$$

ansatz: $\varphi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$$-\frac{\hbar^2}{2M}\Big[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}_{\nabla^2}\Big]RY = \left(E-V(r)\right)RY$$

But

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)}_{\frac{i}{\hbar}\hat{p}_r}\underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)}_{\frac{i}{\hbar}\hat{p}_r}$$

$$\equiv -\frac{1}{\hbar^2}\hat{p}_r^2$$

definitions of $\hat{L}^2$, $\hat{p}_r$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \equiv -\frac{\hat{L}^2}{\hbar^2}$$

$$\left[\frac{\hat{p}_r^2}{2M} + \frac{\hat{L}^2}{2Mr^2}\right]RY = \left(E-V(r)\right)RY$$

or

$$\frac{1}{R}\frac{\hat{p}_r^2}{2M}R(r) + \frac{1}{Y}\frac{\hat{L}^2 Y}{2Mr^2} = E - V(r)$$

$$\Downarrow$$

$$\begin{cases} \hat{L}^2\,Y(\theta,\phi) = \hbar^2\ell(\ell+1)\,Y(\theta,\phi) & \text{angular equation} \\ \left[\frac{\hat{p}_r^2}{2M} + V(r) + \frac{\hbar^2\,\ell(\ell+1)}{2Mr^2}\right]R(r) = E\,R(r) & \text{radial equation} \end{cases}$$