# Matter wave Equation

- What we know:

**Einstein**

$$E = hf = \left(\frac{h}{2\pi}\right)(2\pi f) = \hbar\omega$$

**deBroglie**

$$p = \frac{h}{\lambda} = \left(\frac{h}{2\pi}\right)\left(\frac{2\pi}{\lambda}\right) = \hbar K$$

- plane wave solutions $\Psi(x,t) \cong e^{i(Kx-\omega t)}$
- Correspondence w/ classical relation $E = p^2/2m + V$ (total = kinetic + potential)

$\Rightarrow$ Interpret $E$, $p$ (and all other observables) as linear operators

Measurement yields eigenvalue of appropriate operator (more on this later)

If plane wave is eigenfunction of energy and momentum,

$$\overleftrightarrow{E}\Psi = \hbar\omega\Psi \text{ so } \overleftrightarrow{E} = i\hbar\frac{\partial}{\partial t} \quad / \quad \overleftrightarrow{p}\Psi = \hbar K\Psi \text{ so } \overleftrightarrow{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

then $\overleftrightarrow{E}\Psi = \left(\frac{\overleftrightarrow{p}^2}{2m} + \overleftrightarrow{V}\right)\Psi$ becomes

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)$$

the "time-dependent Schrodinger equation"

## Simplest case: $V(x)=0$ "Free particle"

Solution: plane wave $\Psi(x,t) = e^{i(kx-\omega t)}$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

$$\frac{\hbar^2 k^2}{2m}\Psi = \hbar\omega\Psi \quad \Rightarrow \quad \omega = \frac{\hbar k^2}{2m}$$ "dispersion relation"

c.f. Classical wave: $\omega = kc$ $(\lambda f = c)$ $V_{phase} = \frac{\omega}{k} = c$ $V_{group} = \frac{d\omega}{dk} = c$

For matter waves, $V_{phase} = \frac{\omega}{k} = \frac{\hbar k}{2m}$, $V_{group} = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} = \frac{mv}{m} = v$

Corresponds to classical velocity!

## What is $\Psi$?

"Born Rule" / "Copenhagen Interpretation"

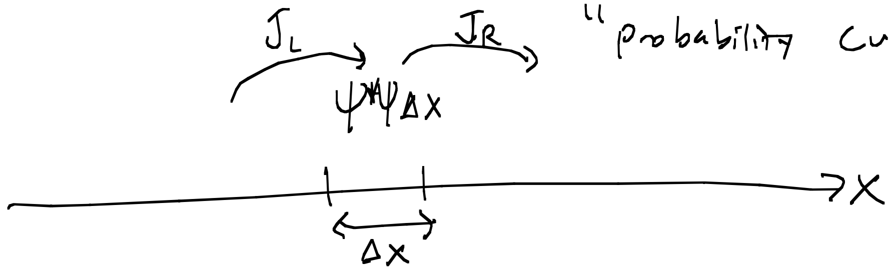
$|\Psi|^2 = \Psi^* \Psi$ is a probability density

Consequences:

- Normalization: $\displaystyle\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1$

- Probability is conserved!

"probability current"




$$\frac{\partial}{\partial t}\left(\Psi^*\Psi \Delta x\right) = J_L - J_R \longrightarrow \frac{\partial}{\partial t}\Psi^*\Psi = \frac{J_L - J_R}{\Delta x}$$

In the continuous limit $\Delta x \to 0$, $\dfrac{\partial}{\partial t}\Psi^*\Psi = -\dfrac{\partial J}{\partial x}$ "Continuity equation"

What is Probability current $J$?

$$\frac{\partial}{\partial t}(\psi^*\psi) = \left(\frac{\partial}{\partial t}\psi^*\right)\psi + \psi^*\left(\frac{\partial}{\partial t}\psi\right)$$

From TDSE: $\frac{\partial \psi}{\partial t} = -\frac{\hbar}{2mi}\frac{\partial^2\psi}{\partial x^2} + \frac{V}{i\hbar}\psi$ and $\frac{\partial \psi^*}{\partial t} = \frac{\hbar}{2mi}\frac{\partial^2\psi^*}{\partial x^2} - \frac{V}{i\hbar}\psi^*$

$$\frac{\partial}{\partial t}(\psi^*\psi) = \frac{\hbar}{2mi}\frac{\partial^2\psi^*}{\partial x^2}\psi - \cancel{\frac{V}{i\hbar}\psi^*\psi} - \psi^*\frac{\hbar}{2mi}\frac{\partial^2\psi}{\partial x^2} + \cancel{\psi^*\frac{V}{i\hbar}\psi}$$

$$= -\frac{\partial}{\partial x}\left[\frac{\hbar}{2mi}\left(\psi^*\frac{\partial}{\partial x}\psi - \frac{\partial\psi^*}{\partial x}\psi\right)\right] = -\frac{\partial J}{\partial x}$$

So $$J = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial}{\partial x}\psi - \frac{\partial\psi^*}{\partial x}\psi\right) = \mathrm{Im}\left\{\frac{\hbar}{m}\psi^*\frac{\partial\psi}{\partial x}\right\}$$

Example: free particle plane wave $\psi = e^{i(kx-\omega t)}$

$$J = \mathrm{Im}\left\{\frac{\hbar}{m}e^{-i(kx-\omega t)}\,ik\,e^{i(kx-\omega t)}\right\} = \psi^*\psi\,\frac{\hbar k}{m} \sim \text{density}\cdot\text{velocity}$$

c.f. classical particle flux!

## Comparison with energy flux carried by classical E+M wave

"Poynting" energy flux : $\vec{j} = \frac{1}{2} \mathrm{Re}\left\{ \vec{\mathcal{E}} \times \vec{B}^* \right\}$

Faraday's Law : $\vec{\nabla} \times \vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t} = i\omega \vec{B}$ when time dependence of $\vec{B}$ is $\propto e^{-i\omega t}$ as ~~in~~ a propagating wave. Then, $B^* = -\frac{1}{i\omega} \vec{\nabla} \times \vec{\mathcal{E}}^*$

If $\vec{\mathcal{E}} = \mathcal{E}_z \hat{z}$ as in a wave propagating in the $\hat{x}$-$\hat{y}$ plane,

$$\nabla \times \mathcal{E}^* = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \mathcal{E}_z^* \end{vmatrix} = \hat{x} \frac{\partial \mathcal{E}_z^*}{\partial y} - \hat{y} \frac{\partial \mathcal{E}_z^*}{\partial x} = -(\hat{z} \times \hat{y}) \frac{\partial \mathcal{E}_z^*}{\partial y} - (\hat{z} \times \hat{x}) \frac{\partial \mathcal{E}_z^*}{\partial x} = -\hat{z} \times \vec{\nabla} \mathcal{E}_z^*$$

Therefore, $B^* = -\frac{1}{i\omega} \vec{\nabla} \times \vec{\mathcal{E}}^* = +\frac{1}{i\omega} \hat{z} \times \vec{\nabla} \mathcal{E}_z^*$ and

$$j = \frac{1}{2} \mathrm{Re}\left\{ \vec{\mathcal{E}} \times \frac{1}{i\omega} \hat{z} \times \vec{\nabla} \mathcal{E}_z^* \right\}$$

Since $\vec{A}\times\vec{B}\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$, this can be written

$$= -\frac{1}{2w}\,\mathrm{Re}\left\{ i\left(\hat{z}\left(\hat{z}\mathcal{E}_z, \vec{\nabla}\mathcal{E}_z^*\right) - \vec{\nabla}\mathcal{E}_z^*\left(\hat{z}\mathcal{E}_z\cdot\hat{z}\right)\right)\right\}$$

The first term vanishes because of Gauss' Law:

$\hat{z}\cdot\nabla\mathcal{E}_z^* = \frac{d}{dz}\mathcal{E}_z^*$ but since $\vec{\mathcal{E}} = \mathcal{E}_z\hat{z}$, $\vec{\nabla}\cdot\vec{\mathcal{E}}^* = \frac{\partial}{\partial z}\mathcal{E}_z^* = 0$

Then

$$\vec{j} = \frac{1}{2w}\,\mathrm{Re}\left\{ i\,\mathcal{E}_z\,\vec{\nabla}\vec{\mathcal{E}}_z^* \right\} = \frac{1}{2w}\,\mathrm{Im}\left\{\mathcal{E}_z^*\nabla\mathcal{E}_z\right\} \qquad \text{(energy flux in E+M)}$$

This is mathematically identical to

$$\vec{J} = \frac{\hbar}{m}\,\mathrm{Im}\left\{\psi^*\nabla\psi\right\} \qquad \text{(probability flux in QM)}$$