Calculating eigenvalues and eigenvectors

$$\overleftrightarrow{M}\vec{x} = \lambda\vec{x} = \lambda\overleftrightarrow{I}\vec{x}$$

identity $\overleftrightarrow{I} = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$ (diagonal)

$$\left(\overleftrightarrow{M} - \lambda\overleftrightarrow{I}\right)\vec{x} = 0$$

$$\det\left(\overleftrightarrow{M} - \lambda\overleftrightarrow{I}\right) = 0$$

Example: $\overleftrightarrow{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0$

$\lambda = +1, -1$

eigenvectors:

$\lambda = +1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 = x_2 \quad \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (elements chosen to normalize $|\vec{x}| = 1$)

$\lambda = -1$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 = -x_2 \quad \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

But if eigenvalues are roots of a polynomial, in general they are complex valued. Since eigenvalues have physical meaning in our problem, how can we be certain they are always real valued?

Matrix Symmetry

The key to physically meaningful eigenvalues/eigenvectors is that we will always have a "Hermitian" matrix: One that is equal to its own Hermitian conjugate (transpose / complex conjugate): $\overleftrightarrow{H} = \overleftrightarrow{H}^\dagger$

(Note: $(AB)^\dagger = B^\dagger A^\dagger$)

Proof that eigenvalues of Hermitian matrix are always real:

$$\overleftrightarrow{H}\vec{x} = \lambda\vec{x}$$

$$(\overleftrightarrow{H}\vec{x})^\dagger = (\lambda\vec{x})^\dagger$$

$$\vec{x}^\dagger\overleftrightarrow{H}^\dagger = \vec{x}^\dagger\lambda^*$$

$$\vec{x}^\dagger\overleftrightarrow{H} = \vec{x}^\dagger\lambda^*$$

$$\cancel{\vec{x}^\dagger}\overleftrightarrow{H}\vec{x} = \cancel{\vec{x}^\dagger}\lambda^*\vec{x}$$

$$\overleftrightarrow{H}\vec{x} = \lambda^*\vec{x} \longrightarrow$$ so $\lambda = \lambda^*$ and therefore must be real!

eigenvectors of Hermitian matrices

take two different eigenvector/eigenvalue pairs of hermitian matrix H:

$$\overleftrightarrow{H}\vec{X}_1 = a\vec{X}_1 \qquad \overleftrightarrow{H}\vec{X}_2 = b\vec{X}_2$$

$$\vec{X}_2^{+}\overleftrightarrow{H}\vec{X}_1 = \vec{X}_2^{+} a\vec{X}_1 \qquad \left(\overleftrightarrow{H}\vec{X}_2\right)^{+} = \left(b\vec{X}_2\right)^{+}$$

$$\vec{X}_2^{+}\overleftrightarrow{H}\vec{X}_1 = a\vec{X}_2^{+}\vec{X}_1 \qquad \vec{X}_2^{+}\overleftrightarrow{H}^{+} = \vec{X}_2^{+} b^{*}$$

$$\vec{X}_2^{+}\overleftrightarrow{H}^{+}\vec{X}_1 = \vec{X}_2^{+} b^{*}\vec{X}_1 = b^{*}\vec{X}_2^{+}\vec{X}_1$$

So: $a\vec{X}_2^{+}\vec{X}_1 = b^{*}\vec{X}_2^{+}\vec{X}_1$ Since a, b are real and distinct,

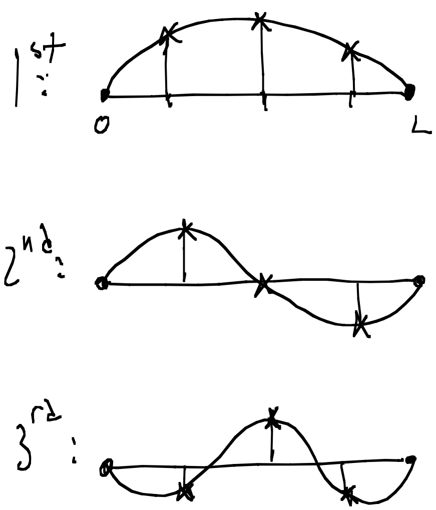
This can only be true if $\vec{X}_2^{+}\vec{X}_1 = 0$ $\left(\vec{X}_2 \cdot \vec{X}_1 = 0\right)$

In other words, the eigenvectors of a Hermitian matrix are orthogonal!

Question: What are the eigenvectors of this matrix?

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

We have seen this matrix before: It was the matrix equivalent of the second derivative from the wave equation! So its eigenvectors are approximations to the modes of a standing wave satisfying $\overleftrightarrow{H}\vec{X} = \lambda \vec{X}$:



Matlab:

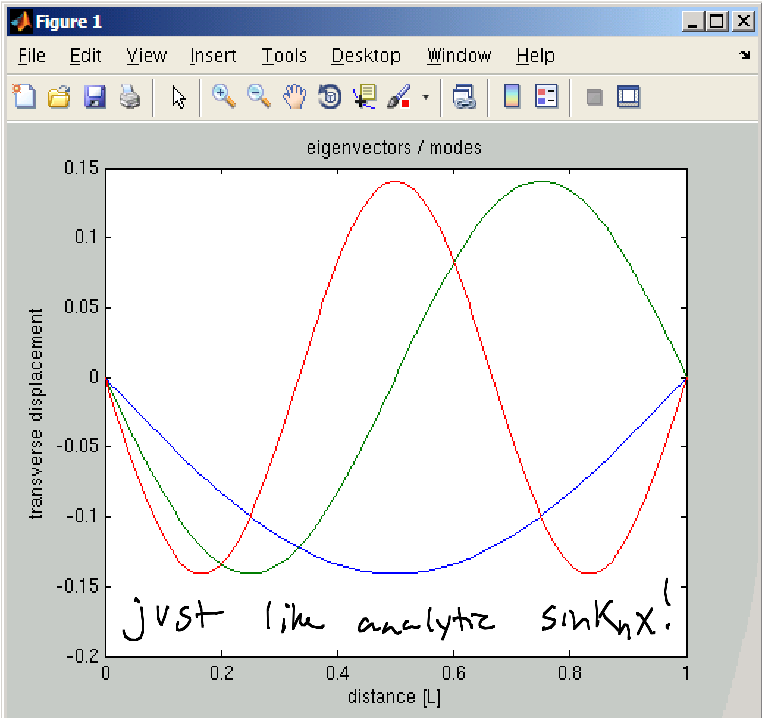
```
>> H=[-2 1 0;1 -2 1; 0 1 -2]

H =

    -2     1     0
     1    -2     1
     0     1    -2

>> [v,d]=eig(H)

v =

    0.5000   -0.7071   -0.5000
   -0.7071    0.0000   -0.7071
    0.5000    0.7071   -0.5000

d=
```

An even better way to define H:

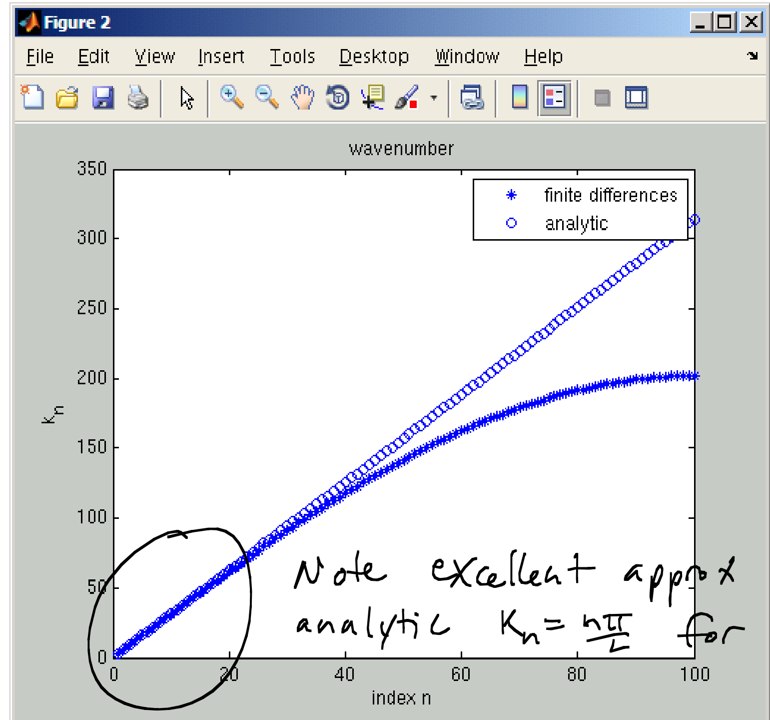
```
>> N=3;H=diag(ones(N-1,1),1)+diag(-2*ones(N,1),0)+diag(ones(N-1,1),-1)

H =

    -2     1     0
     1    -2     1
     0     1    -2
```

Now, with larger values of the variable N, we can generate a much larger H corresponding to smaller Δx and hence better approximation to the analytic sol'ns!

Numerical Solution to Standing wave eigenvalue problem



just like analytic $\sin k_n x$!



Note excellent approx to analytic $k_n = \frac{n\pi}{L}$ for low n.

Matlab hints:

[v, d] = eig(H) returns eigenvectors as columns in the matrix v: The n^{th} eigenvector is v(:,n). The corresponding eigenvalues are given as diagonal elements of matrix d. They can be recovered into an array via diag(d).