Boundary Conditions: Eigenvalue / Eigenfunction problem

example: Classical wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

$x=0$ $\quad f(x=0,t)=0$

$x=L$ $\quad f(x=L,t)=0$

Separation of variables: $f(x,t) = X(x)\,T(t)$

$$\frac{X''T}{XT} = \frac{\frac{1}{c^2}X\ddot{T}}{XT}$$

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -k^2$$

$X'' = -k^2 X$ and $\ddot{T} = -k^2c^2 T$

$$X(x) = A\cos kx + B\sin kx$$

Apply Boundary Conditions:

$X(x=0) = 0 = A$

$X(x=L) = B\sin kL = 0$

So $kL = n\pi$ $\quad n = 1, 2, 3, \ldots$

$k = \frac{n\pi}{L}$ (discrete) "eigenvalues"

Then, our general solution can be written

$$f(x,t) = \sqrt{\frac{2}{L}}\sum_n \sin k_n x \cdot \left[C_n \sin\omega_n t + D_n\cos\omega_n t\right]$$

where $\omega_n = k_n c$

Applying initial conditions

With $f(x,t=0)$ specified, we can determine the coefficients C_n, D_n.

by using orthonormality of the basis functions ("modes"),

e.g. $$f(x,t=0) = \sqrt{\frac{2}{L}} \sum_n D_n \sin k_n x$$

Take inner product w/ orthonormal basis functions $u_m(x) = \sqrt{\frac{2}{L}} \sin k_m x$:

$$\sqrt{\frac{2}{L}} \int_0^L \sin k_m x \, f(x,t=0) \, dx = \int_0^L \frac{2}{L} \sin k_m x \left(\sum_n D_n \sin k_n x \right) dx$$

Reverse order of sum/integral $$= \sum_n D_n \left(\int_0^L \frac{2}{L} \sin k_m x \sin k_n x \right) dx$$

By orthonormality, $$= \sum_n D_n \, \delta_{nm} = D_m$$

So, $$D_n = \int_0^L u_n^*(x) \, f(x,t=0) \, dx$$ look familiar?

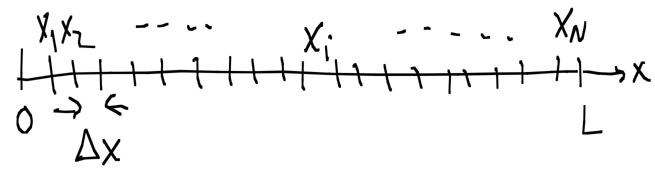
Numerical Solution of $\frac{d^2 \mathbb{X}(x)}{dx^2} = -k^2 \mathbb{X}(x)$

Use symmetric definition of derivative $\frac{d}{dx}\mathbb{X}(x) = \lim_{\Delta x \to 0} \frac{\mathbb{X}(x + \frac{\Delta x}{2}) - \mathbb{X}(x - \frac{\Delta x}{2})}{\Delta x}$

Then, $$\frac{d^2}{dx^2}\mathbb{X}(x) = \lim_{\Delta x \to 0} \frac{\frac{\mathbb{X}(x+\Delta x) - \mathbb{X}(x)}{\Delta x} - \frac{\mathbb{X}(x) - \mathbb{X}(x-\Delta x)}{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\mathbb{X}(x-\Delta x) - 2\mathbb{X}(x) + \mathbb{X}(x+\Delta x)}{\Delta x^2}$$

Now, instead of passing to limit $\Delta x \to 0$ to recover continuous differential operator, use "Finite differences": Approximate derivatives by using nonzero value of Δx

i.e. <u>discretize</u> continuous variable x : N segments from $x=0$ to $x=L$:



Then,

$$\frac{d^2\mathbb{X}(x_i)}{dx^2} \sim \frac{\mathbb{X}(x_{i-1}) - 2\mathbb{X}(x_i) + \mathbb{X}(x_{i+1})}{\Delta x^2} \equiv \boxed{\frac{\mathbb{X}_{i-1} - 2\mathbb{X}_i + \mathbb{X}_{i+1}}{\Delta x^2} = -k^2 \mathbb{X}_i}$$

From differential Equation to a linear system of algebraic eqns

Evaluate at every value of x_i using finite differences approximation:

$$i=1: \quad \frac{1}{\Delta x^2}\left(-2X_1 + X_2 \right) = -k^2 X_1$$

$$i=2: \quad \frac{1}{\Delta x^2}\left(X_1 - 2X_2 + X_3 \right) = -k^2 X_2$$

$$i=3: \quad \frac{1}{\Delta x^2}\left(X_2 - 2X_3 + X_4 \right) = -k^2 X_3$$

$$\vdots$$

$$i=N: \quad \frac{1}{\Delta x^2}\left(X_{N-1} - 2X_N \right) = -k^2 X_N$$

Equivalent to:

$$\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & \\ 1 & -2 & 1 & 0 & \cdots & \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ & & & \ddots & & \\ & \cdots & & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \end{bmatrix} = -k^2 \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \end{bmatrix}$$

a matrix eigenvalue problem!