Fourier Transform of Gaussian: $\left(f(x) = e^{-ax^2}, \quad \Delta x = \frac{1}{\sqrt{2a}} \right)$

$$A(k) = \int_{-\infty}^{\infty} e^{-ax^2} \frac{e^{-ikx}}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x^2 + \frac{ikx}{a}\right)} dx$$

Complete the square: $$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x + \frac{ik}{2a}\right)^2} e^{-\frac{k^2}{4a}} dx$$

Variable transformation: $z = \sqrt{a}\left(x + \frac{ik}{2a}\right), \quad dx = \frac{dz}{\sqrt{a}}$

$$= \frac{1}{\sqrt{2\pi a}} e^{-\frac{k^2}{4a}} \overbrace{\int_{-\infty}^{\infty} e^{-z^2} dz}^{\sqrt{\pi}} = \frac{1}{\sqrt{2a}} e^{-\frac{k^2}{4a}}$$

So, the F.T. of gaussian in x is a gaussian in k!

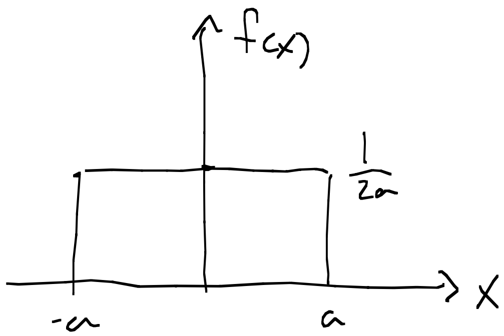
"Heisenberg Uncertainty Principle"

Note the reciprocal relationship between the standard deviation "uncertainty" $\Delta x = \frac{1}{\sqrt{2a}}$ of $f(x)$ and its transform $\Delta k = \sqrt{2a}$ of $A(k)$. A large Δx means a small Δk and vice versa. Therefore,

$$\Delta x \cdot \Delta k = 1$$

The gaussian minimizes this product, so in general $\Delta x \Delta k \geq 1$. Physicists call this the "Heisenberg Uncertainty principle" in the context of QM, but clearly it is only a consequence of the mathematics of the Fourier Transform (and its wavelike basis fns)!

Generality of "Heisenberg uncertainty principle"



$\Delta x = a$

$$A(k) = \int_{-\infty}^{\infty} f(x) \frac{e^{-ikx}}{\sqrt{2\pi}} dx = \int_{-a}^{a} \frac{1}{2a} \frac{e^{-ikx}}{\sqrt{2\pi}} dx = \frac{1}{2a\sqrt{2\pi}} \frac{e^{-ikx}}{-ik}\Big|_{-a}^{a}$$

$$= \frac{1}{2a\sqrt{2\pi}} \frac{e^{-ika} - e^{ika}}{-ik} = \frac{1}{ka\sqrt{2\pi}} \frac{(e^{ika} - e^{-ika})}{2i} = \frac{1}{\sqrt{2\pi}} \frac{\sin ka}{ka}$$

$$\propto \operatorname{sinc}(ka)$$

$A(k) \propto \operatorname{sinc}(ka)$

$\Delta k = \frac{\pi}{a}$

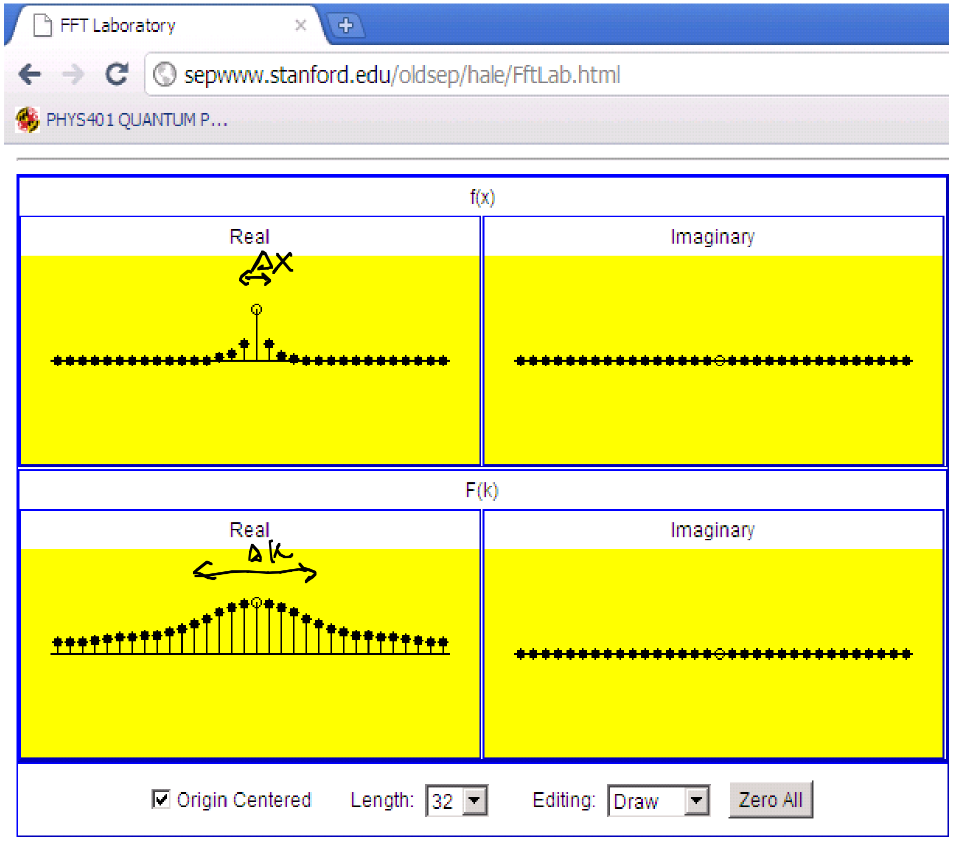
$\Delta x = a, \quad \Delta k = \frac{\pi}{a} \; ! \quad \Delta x \Delta k = \pi > 1$

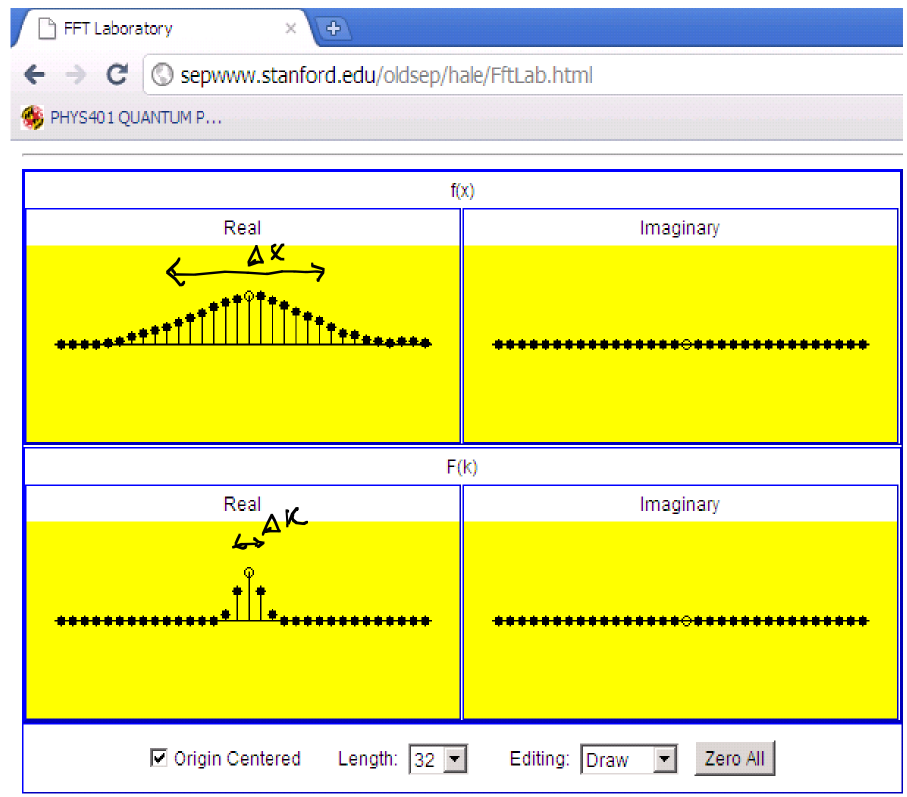
So reciprocal relationship holds!

Experimenting w/ Fourier Transforms

$f(x) =$

$A(k) =$





Note reciprocal relationship between Δx and Δk!