Spin in magnetic field

$$\vec{S} = S_x \hat{x} + S_y \hat{y} + S_z \hat{z} = \frac{\hbar}{2}\left(\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}\right), \quad \langle S^2 \rangle = \hbar^2 s(s+1), \quad s = 1/2$$

σ's are 2x2 Pauli matrices.

$$\langle S_z \rangle = m_s \hbar \qquad m_s = \pm\frac{1}{2}$$

Hamiltonian: $H = 2\frac{\mu_B}{\hbar}\vec{S}\cdot\vec{B} = \mu_B \sigma_z B_z \qquad (\vec{B} = B_z \hat{z})$

"Lorentz boost" → "Thomas factor"

"electron g-factor"

$$H = \begin{bmatrix} \mu_B B_z & 0 \\ 0 & -\mu_B B_z \end{bmatrix} \xrightarrow[\text{eigenvectors}]{\text{eigenvalues}} \begin{array}{ll} E = +\mu_B B_z, \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{"up"} = |\uparrow\rangle \\ E = -\mu_B B_z, \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{"down"} = |\downarrow\rangle \end{array}$$

$$= \overset{2}{g}\,\mu_B B_z m_s$$

State evolution

Example: $|\psi(0)\rangle$ is eigenvector of $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} +\frac{\hbar}{2},\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle \\ -\frac{\hbar}{2},\ \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle \end{array}$

Now, apply $\vec{B} = B_z \hat{z}$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} : \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\frac{\mu_B B_z t}{\hbar}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{+i\frac{\mu_B B_z t}{\hbar}} \right)$$

what is $\langle S_x(t)\rangle$?

$$\langle +|S_x|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{+i\frac{\mu_B B_z t}{\hbar}} & e^{-i\frac{\mu_B B_z t}{\hbar}} \end{bmatrix} \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\frac{\mu_B B_z t}{\hbar}} \\ e^{+i\frac{\mu_B B_z t}{\hbar}} \end{bmatrix}$$

$$= \frac{\hbar}{4}\left(e^{2i\frac{\mu_B B_z t}{\hbar}} + e^{-2i\frac{\mu_B B_z t}{\hbar}} \right) = \frac{\hbar}{2}\cos\frac{2\mu_B B_z t}{\hbar}$$

Similarly, $\langle +|S_y|+\rangle = \hbar/2 \sin\frac{2\mu_B B_z t}{\hbar}$

precession!

Adding orbital and spin angular momentum

$$H = \frac{\mu_B}{\hbar}\vec{L}\cdot\vec{B} + 2\frac{\mu_B}{\hbar}\vec{S}\cdot\vec{B} = \frac{\mu_B}{\hbar}(\vec{L}+2\vec{S})\cdot\vec{B}$$

Only $\vec{J} = \vec{L}+\vec{S}$ is conserved, not $\vec{L}+2\vec{S}$. Does $E = \text{"}g\text{"}\mu_B B_z m_J$??

$$H = \frac{\mu_B}{\hbar}\frac{(\vec{L}+2\vec{S})\cdot\vec{J}}{|J|}\cdot\frac{\vec{B}\cdot\vec{J}}{|J|} = \frac{\mu_B}{\hbar}\frac{(\vec{L}+2\vec{S})(\vec{L}+\vec{S})\;J_zB_z}{J^2}$$

$$= \frac{\mu_B}{\hbar}\frac{\left(L^2+2S^2+3\vec{S}\cdot\vec{L}\right)J_zB_z}{J^2}$$

$L^2 = \hbar^2 l(l+1)$, $l = 0,1,2..$

$S^2 = \hbar^2 s(s+1) = \frac{3}{4}\hbar^2$

$J^2 = \hbar^2 j(j+1)$, $j = l-s,\ l+s$

What is $\vec{S}\cdot\vec{L}$?

$$J^2 = (\vec{L}+\vec{S})^2 = L^2+S^2+2\vec{S}\cdot\vec{L}$$

So

$$S\cdot L = \frac{J^2-L^2-S^2}{2}$$

Landé g-factor

$$E = \frac{\mu_B}{\hbar} \frac{\left(L^2 + 2S^2 + \frac{3}{2}\left(J^2 - L^2 - S^2\right)\right)}{J^2} \hbar m_J B_z$$

$$= \mu_B \frac{\frac{3}{2}J^2 - \frac{1}{2}L^2 + \frac{1}{2}S^2}{J^2} m_J B_z = \left(1 + \frac{J^2 - L^2 + S^2}{2J^2}\right) \mu_B m_J B_z$$

$$= g_L \mu_B m_J B_z$$

where g_L ("Landé g-factor) is given by

$$g_L = 1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

Landé g-factor examples

$l=0$

$j = s = \frac{1}{2}$

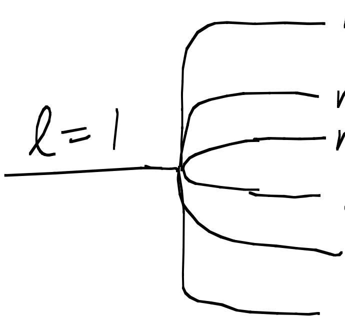
$$g_L = 1 + \frac{\frac{1}{2}\left(\frac{3}{2}\right) - 0(0+1) + \frac{1}{2}\left(\frac{3}{2}\right)}{2\,\frac{1}{2}\left(\frac{3}{2}\right)} = 2$$

$l=1$

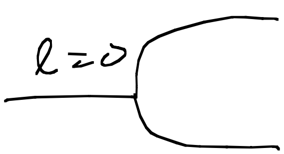
$j = \frac{1}{2}$ or $\frac{3}{2}$

$$g_L = \begin{cases} 1 + \dfrac{\frac{1}{2}\left(\frac{3}{2}\right) - 1(2) + \frac{1}{2}\left(\frac{3}{2}\right)}{2\cdot\frac{1}{2}\left(\frac{3}{2}\right)} = 1 + \dfrac{\frac{3}{4} - \frac{8}{4} + \frac{3}{4}}{\frac{6}{4}} = \dfrac{2}{3} \quad \left(j = \frac{1}{2}\right) \\ \text{or} \\ 1 + \dfrac{\frac{3}{2}\left(\frac{5}{2}\right) - 1(2) + \frac{1}{2}\left(\frac{3}{2}\right)}{2\,\frac{3}{2}\left(\frac{5}{2}\right)} = 1 + \dfrac{\frac{15}{4} - \frac{8}{4} + \frac{3}{4}}{\frac{30}{4}} = \dfrac{4}{3} \quad \left(j = \frac{3}{2}\right) \end{cases}$$

Anomalous Zeeman Effect



m_j	j	g_L
$m_j = +\frac{3}{2}$	$3/2$	$4/3$
$m_j = +\frac{1}{2}$	$3/2$	$4/3$
$m_j = +\frac{1}{2}$	$1/2$	$2/3$
$m_j = -\frac{1}{2}$	$1/2$	$2/3$
$m_j = -\frac{1}{2}$	$3/2$	$4/3$
$m_j = -\frac{3}{2}$	$3/2$	$4/3$



$m_j = \frac{1}{2}$

$m_j = -\frac{1}{2}$ $\quad (g_L = 2)$

$j = \frac{1}{2}$

$j = 3/2 \rightarrow j = \frac{1}{2}$ transitions:

$\Delta E \propto$

$(g_L m_j)^i - (g_L m_j)^f$

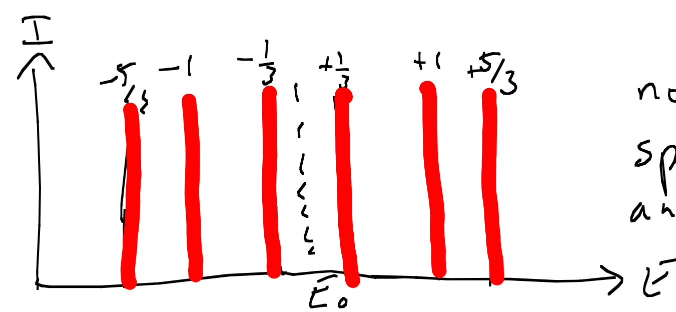
$2 - 1 \quad = +1$

$\frac{2}{3} - 1,\ \frac{2}{3} + 1 \quad = -\frac{1}{3},\ \frac{5}{3}$

$-\frac{2}{3} + 1,\ -\frac{2}{3} - 1 = \frac{1}{3},\ -\frac{5}{3}$

$-2 + 1 \quad = -1$

Spectroscopy in B-field:



note 6 spectral lines, and none @ E_0!

All because g_L is different for different states!

In terms of matrices

Example: $l=1$, $s=\frac{1}{2}$

Then (without proof)

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

what is $J^2 = L^2 + S^2 + 2\vec{S}\cdot\vec{L}$? a 6×6 matrix!!

$$\begin{aligned} = &(I_2 \otimes L_x)^2 + (S_x \otimes I_3)^2 + 2 S_x \otimes L_x \\ &+ (I_2 \otimes L_y)^2 + (S_y \otimes I_3)^2 + 2 S_y \otimes L_y \\ &+ (I_2 \otimes L_z)^2 + (S_z \otimes I_3)^2 + 2 S_z \otimes L_z \end{aligned}$$

Kronecker product merges the 2×2 space w/ 3×3 space to create a larger 6×6 space!

Numerical Calculation

```
Sx=1/2*[0 1;1 0];
Sy=1/2*[0 -i ; i 0];
Sz=1/2*[1 0; 0 -1];
Lx=1/sqrt(2)*[0 1 0; 1 0 1; 0 1 0];
Ly=1/sqrt(2)*[0 -i 0; i 0 -i; 0 i 0];
Lz=[1 0 0; 0 0 0 ; 0 0 -1];

Ssqrd=kron(eye(3), Sx)^2+kron(eye(3), Sy)^2+kron(eye(3), Sz)^2;
Lsqrd=kron(Lx, eye(2))^2+kron(Ly, eye(2))^2+kron(Lz, eye(2))^2;
SdotL=kron(Lx,Sx)+kron(Ly,Sy)+kron(Lz,Sz);

Jsqrd=Ssqrd+Lsqrd+2*SdotL;

[v,d]=eig(Jsqrd);
diag(d)

gL=(Lsqrd+2*Ssqrd+3*SdotL)/Jsqrd;
eig(gL)
```

$$\frac{J^2}{\hbar^2} = \begin{bmatrix} 3.75000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 1.75000 & 1.41421 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 1.41421 & 2.75000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 2.75000 & 1.41421 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 1.41421 & 1.75000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 3.75000 \end{bmatrix}$$

eigenvalues of $\frac{J^2}{\hbar^2}$:

$$\left.\begin{matrix} 0.75000 \\ 0.75000 \end{matrix}\right\} = \frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right)$$

$$\left.\begin{matrix} 3.75000 \\ 3.75000 \\ 3.75000 \\ 3.75000 \end{matrix}\right\} = \frac{15}{4} = \frac{3}{2}\left(\frac{3}{2}+1\right)$$

$$\langle g_L \rangle = \begin{matrix} 0.66667 \\ 1.33333 \\ 1.33333 \\ 0.66667 \\ 1.33333 \\ 1.33333 \end{matrix}$$

i.e. $\frac{2}{3}$ or $\frac{4}{3}$, Same as calculation w/ $j = \ell \pm s$ and

$$1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}$$