Second derivative of $u(\rho)$

$$u(\rho) = \rho^{\ell+1}\left(e^{-\rho} v(\rho)\right) = f \cdot g$$

$$(f \cdot g)' = f'g + g'f$$

$$(f \cdot g)'' = f''g + 2g'f' + g''f$$

$$f' = (\ell+1)\rho^{\ell}, \quad f'' = (\ell+1)\ell\rho^{\ell-1}$$

$$g' = v'e^{-\rho} - e^{-\rho}v, \quad g'' = e^{-\rho}v - 2e^{-\rho}v' + e^{-\rho}v''$$

$$u'' = \ell(\ell+1)\rho^{\ell-1}e^{-\rho}v + 2(\ell+1)\rho^{\ell}\left(v'e^{-\rho} - e^{-\rho}v\right) + \rho^{\ell+1}\left(e^{-\rho}v - 2e^{-\rho}v' + e^{-\rho}v''\right)$$

$$= \rho^{\ell}e^{-\rho}\left[\rho v'' + \left(2(\ell+1) - 2\rho\right)v' + \left(\frac{\ell(\ell+1)}{\rho} - 2(\ell+1) + \rho\right)v\right]$$

Equation for $v(\rho)$

$$u'' + \left[\frac{\rho_0}{\rho} - \frac{\ell(\ell+1)}{\rho^2} - 1\right]u = 0 \,, \qquad u = \rho^{\ell+1}e^{-\rho}v = \rho^{\ell}e^{-\rho}(\rho v)$$

$$\rho^{\ell}e^{-\rho}\Big[\rho v'' + \big(2(\ell+1) - 2\rho\big)v' + \left(\cancel{\frac{\ell(\ell+1)}{\rho}} - 2(\ell+1) + \cancel{\rho}\right)v + \left(\rho_0 - \cancel{\frac{\ell(\ell+1)}{\rho}} - \cancel{\rho}\right)v\Big] = 0$$

$$\rho v'' + 2(\ell+1-\rho)v' + \big(\rho_0 - 2(\ell+1)\big)v = 0$$

Due to non-constant coefs, must solve via "brute-force"!

"Brute force" solution

$$V(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$

$$\frac{dv(\rho)}{d\rho} = \sum_{j=0}^{\infty} j c_j \rho^{j-1} = \sum_{j=0}^{\infty} (j+1) c_{j+1} \rho^j$$

$$\frac{d^2 v(\rho)}{d\rho^2} = \sum_{j=0}^{\infty} j(j+1) c_{j+1} \rho^{j-1}$$

$$\rho v'' + 2(\ell+1-\rho) v' + (\rho_0 - 2(\ell+1)) v = 0$$ now becomes:

$$\sum_{j=0}^{\infty} \left[j(j+1) c_{j+1} + 2(\ell+1)(j+1) c_{j+1} - 2j c_j + (\rho_0 - 2(\ell+1)) c_j \right] \rho^j = 0$$

only if all coefs are zero:

$$j(j+1) c_{j+1} + 2(\ell+1)(j+1) c_{j+1} - 2j c_j + (\rho_0 - 2(\ell+1)) c_j = 0$$

Recursion Relation

$$c_{j+1} = \frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} c_j$$

For large j, $c_{j+1} \sim \frac{2}{j+1} c_j \longrightarrow c_j = \frac{2^j}{j!} c_0$

Then, $v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j = \sum_{j=0}^{\infty} c_0 \frac{(2\rho)^j}{j!} = c_0 e^{2\rho}$

So $u(\rho) = \rho^{l+1} e^{-\rho} v(\rho) = c_0 \rho^{l+1} e^{\rho}$ Not normalizable!

So series must terminate with $c_{j_{max}+1} = 0 \rightarrow v(\rho)$ is a polynomial!

principle quantum number

$$2(j_{max}+\ell+1)-\rho_0=0 \quad \text{for some } j_{max}$$

Since integers $j \geq 0$ and $\ell \geq 0$, values of $j_{max}+\ell+1 = 1, 2, 3, \dots \equiv n$

"principle quantum number"

This constrains possible values of $\ell = n - j_{max} - 1 \geq 0$

for $n=1$, $j_{max}=0 \longrightarrow \ell=0$

for $n=2$, $j_{max}=0,1 \longrightarrow \ell=1,0$

for $n=3$, $j_{max}=0,1,2 \longrightarrow \ell=2,1,0$

$\vdots$

$\longrightarrow \ell = n-1, n-2, \dots, 1, 0$