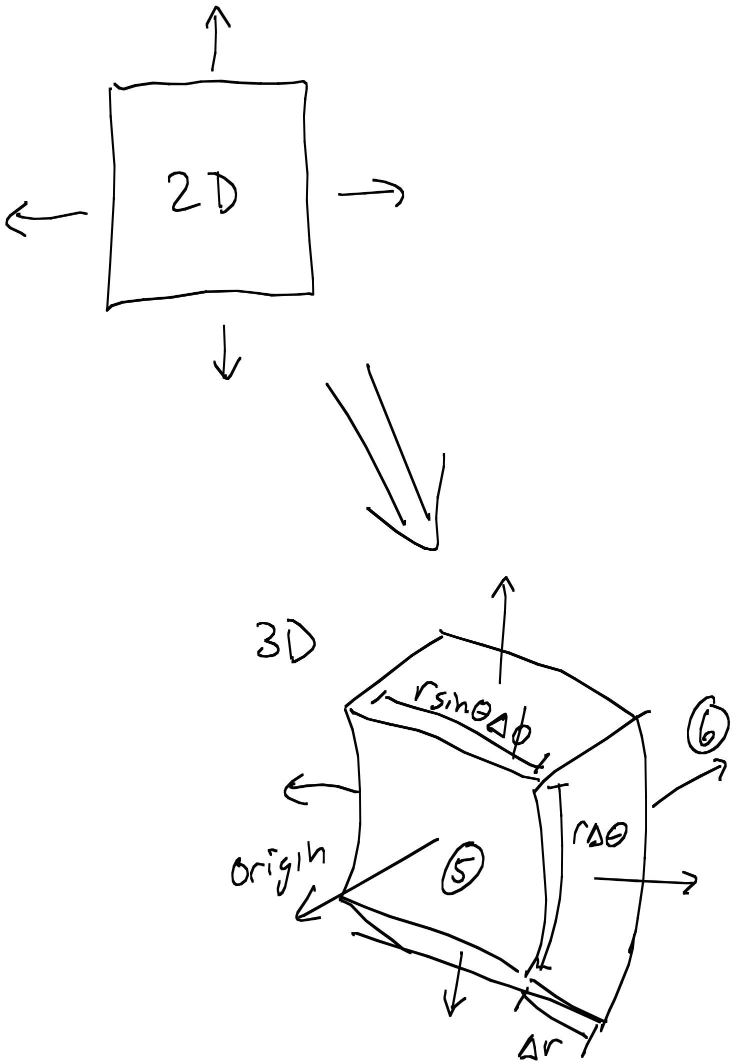
Extension to 3D



Radial Force : 6 − 5

5 $T r^2 \sin\theta \Delta\phi \Delta\theta \dfrac{df(r,\theta,\phi)}{dr}$

6 $T (r+\Delta r)^2 \sin\theta \Delta\phi \Delta\theta \dfrac{df(r+\Delta r,\theta,\phi)}{dr}$

$$\approx T\left(r^2 + 2r\Delta r\right)\sin\theta\Delta\phi\Delta\theta\left(\frac{d}{dr}\left(f(r,\theta,\phi) + \Delta r\frac{df}{dr}\right)\right)$$

$$= T\sin\theta\Delta\phi\Delta\theta\left(r^2\frac{df}{dr} + \left(r^2\Delta r\frac{d^2f}{dr^2} + 2r\Delta r\frac{df}{dr}\right)\right)$$

$$= T\sin\theta\Delta\phi\Delta\theta\left(r^2\frac{df}{dr} + \frac{d}{dr}\left(r^2\frac{df}{dr}\right)\Delta r\right)$$

$$6 - 5 = T\sin\theta\Delta\phi\Delta\theta\frac{d}{dr}\left(r^2\frac{df}{dr}\right)\Delta r$$

Newton's 2nd Law in 3D

$$\frac{T}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\cancel{\Delta\theta}\cancel{\Delta\phi}\cancel{\Delta r} + T\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\cancel{\Delta\theta}\cancel{\Delta\phi}\cancel{\Delta r} + T\sin\theta\cancel{\Delta\theta}\cancel{\Delta\phi}\cancel{\Delta r}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$$

$$= \rho r^2\sin\theta\,\cancel{\Delta\theta}\cancel{\Delta\phi}\cancel{\Delta r}\frac{\partial^2 f}{\partial t^2}$$

$$\underbrace{\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}_{\nabla^2_{3D}\text{ in spheical curvilinear coords}} f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \qquad \left(c^2 \equiv \frac{T}{\rho}\right)$$

Time-independent Schrodinger Eqn in 3D spherical coordinates

1-D : $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$

3-D : $-\frac{\hbar^2}{2m}\nabla_{3D}^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$

Separation of variables : $\psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$

$$-\frac{\hbar^2}{2m}\left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R\right)Y + \left(\nabla_{2D}^2 Y\right)R\right] + V(r)RY = ERY$$

$$\frac{\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) - \frac{2mr^2}{\hbar^2}(V(r)-E)R}{R} = \underbrace{\frac{-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right)}{Y} = \lambda}_{\text{Solutions are spherical harmonics } Y_m^\ell(\theta,\phi)!}$$

This only works if $V(r,\theta,\phi) \to V(r)$! Conserved eigenvalue λ related to angular momentum

Radial Equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(V(r)-E\right)R = \lambda R$$

define $u = rR(r) \longrightarrow R = \frac{u}{r}$ $\qquad \frac{\partial R}{\partial r} = \frac{\partial}{\partial r}\frac{u}{r} = \frac{r\frac{\partial u}{\partial r} - u}{r^2}$

Then $\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} - u\right) = r\frac{\partial^2 u}{\partial r^2} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial r}}$

$$r\frac{\partial^2 u}{\partial r^2} - \frac{2mr}{\hbar^2}\left(V(r)-E\right)u = \lambda \frac{u}{r}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + \left(V(r)-E\right)u = -\frac{\hbar^2}{2m}\frac{\lambda u}{r^2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{dr^2} + \left(V(r) + \frac{\hbar^2 \lambda}{2mr^2}\right)u = Eu$$

centrifugal potential, classically $\frac{L^2}{2mr^2}$

so $L^2 \rightarrow \hbar^2 \ell(\ell+1)$