Separation of Variables

$$\frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) = \frac{r^2}{c^2}\frac{\partial^2 f}{\partial t^2}$$

$f(\theta,\phi,t) = Y(\theta,\phi)\,T(t)$

$$\frac{\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{Y} = \frac{\frac{r^2}{c^2}\frac{\partial^2 T}{\partial t^2}}{T} = -\lambda \quad \text{(unitless)}$$

$T(t)$: $\quad \frac{\partial^2 T}{\partial t^2} = -\lambda\frac{c^2}{r^2}T \quad \longrightarrow \quad T(t) = A_{\pm}e^{\pm i\frac{c}{r}\sqrt{\lambda}t}$

θ and ϕ:

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) = -\lambda Y$$

$Y(\theta,\phi) = F_\phi(\phi)\,F_\theta(\theta)$:

$$\frac{\frac{\partial^2}{\partial\phi^2}F_\phi}{F_\phi} = \frac{-\sin^2\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_\theta}{\partial\theta}\right) + \lambda F_\theta\right)}{F_\theta} = -m^2$$

Azimuthal equation (for $F_\phi(\phi)$)

$$\frac{\partial^2}{\partial\phi^2} F_\phi = -m^2 F_\phi \quad \longrightarrow \quad F_\phi(\phi) = A e^{im\phi}$$

Apply periodic boundary conditions: $F_\phi(\phi) = F_\phi(\phi + 2\pi)$

$$A e^{im\phi} = A e^{im(\phi+2\pi)} = A e^{im\phi} e^{im2\pi}$$

so $m = 0, \pm 1, \pm 2 \, \ldots$

Equation for polar angle θ

$$-\sin^2\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_\theta}{\partial\theta}\right)+\lambda F_\theta\right) = -m^2 F_\theta$$

If $m=0$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF_\theta}{d\theta}\right) = -\lambda F_\theta$$

If $\lambda=0$, $f_\theta(\theta) =$ Constant

If $\lambda \neq 0$ $\quad F_\theta(\theta) \stackrel{?}{=} \cos\theta$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot(-\sin\theta)\right) \stackrel{?}{=} \lambda\cos\theta$$

$$-\frac{1}{\cancel{\sin\theta}}\cdot 2\cancel{\sin\theta}\cos\theta = -\lambda\cos\theta \longrightarrow \text{only if } \lambda=2!$$

What about higher powers of $\cos\theta$?

$F_\theta \overset{?}{=} \cos^2\theta$?

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\cos^2\theta\right) \overset{?}{=} -\lambda\cos^2\theta$$

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(2\cos\theta\sin^2\theta\right)$$

$$-\frac{1}{\sin\theta}\left(-2\sin^3\theta + 2\cos\theta\cdot 2\sin\theta\cos\theta\right) = 2\sin^2\theta - 4\cos^2\theta$$

$$= 2\sin^2\theta + 2\cos^2\theta - 6\cos^2\theta$$

$$= 2 - 6\cos^2\theta \neq -\lambda\cos^2\theta$$

But $F_\theta(\theta) = \cos^2\theta - \frac{1}{3}$ works, with $\lambda = 6$

So, for $m=0$, $\lambda = 0, 2, 6, (12), \ldots$ $\lambda_\ell = \ell(\ell+1)$, $\ell = 0, 1, 3 \ldots$

Example for $m \neq 0$

$l = 1 \quad (\lambda = 2)$

$$-\sin^2\theta \left(2F_\theta + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta} F_\theta\right)\right) = -m^2 F_\theta$$

$F_\theta \stackrel{?}{=} \sin\theta$? :

$$-\sin^2\theta \left(2\sin\theta + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\sin\theta\right)\right) = -m^2 \sin\theta$$

$$-\sin^2\theta \left(2\sin\theta + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cos\theta\right)\right) =$$

$$-\sin^2\theta \left(2\sin\theta + \frac{1}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right)\right) = -m^2 \sin\theta$$

$$-\sin\theta \left(2\sin^2\theta + \cos^2\theta - \sin^2\theta\right)$$

$$-\sin\theta = -m^2 \sin\theta$$

$$\Longrightarrow \quad m = \pm 1$$

In general, $\quad m = -l, -l+1, \ldots l-1, +l$

Solution to angular equation

$$Y_{\ell m}(\theta,\phi) = F_{\theta}^{\ell m}(\theta) F_{\phi}^{m}(\phi)$$ "spherical harmonics"