Partial differential equations

Sol'ns are functions of <u>more</u> than one variable

Example: $$\frac{\partial}{\partial t} f(x,t) = D \frac{\partial^2}{\partial x^2} f(x,t)$$ "Heat"/ "Diffusion" equation

[Shorthand notation: $\dot{f} = D f''$]

Always try "separation of variables":

guess: $f(x,t) = X(x)\,T(t)$

substitute:

$$\frac{X \dot{T}}{D X T} = \frac{D T X''}{D T X}$$

$$\frac{\dot{T}}{DT} = \frac{X''}{X} = -k^2$$

constant, units distance^{-2}

only depends on t — only depends on x

<u>2nd order ODE:</u>

$$X'' = -k^2 X$$

$$X(x) = B(k) e^{ikx} \quad (-\infty < k < \infty)$$

<u>1st order ODE</u>

$$\dot{T} = -Dk^2 T$$

$$T(t) = C(k) e^{-Dk^2 t}$$

general sol'n:

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{-Dk^2 t} e^{ikx}\, dk$$

Initial conditions:

$$f(x,t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk$$

But how to determine $A(k)$ from $f(x,t=0)$??

Application of Initial Conditions: "Fourier Transform"

Change of basis in an orthonormal function space

Analogy with Euclidean vector space:

Vectors: $\vec{F} = \sum_i A_i \vec{V}_i$ $\quad$ ($\vec{V}_i$'s are "orthonormal basis vectors")

(2D) example: $\vec{F} = \overset{A_1}{2}\underbrace{\hat{x}}_{\vec{V}_1} + \overset{A_2}{3}\underbrace{\hat{y}}_{\vec{V}_2}$

functions: $F(x) = \int_{-\infty}^{\infty} A(k)\, u_k(x)\, dk$ $\quad$ ($u_k(x)$'s are "orthonormal basis functions")

$\left(u_k(x) = \dfrac{e^{ikx}}{\sqrt{2\pi}} \text{ for Fourier Transform} \right)$

Determine Coefficients

Vectors $\vec{F} = \sum_i A_i \vec{v}_i$

"inner product": $\vec{V}_j \cdot \vec{F} = \vec{V}_j \cdot \sum_i A_i \vec{v}_i = \sum_i A_i (\vec{V}_j \cdot \vec{V}_i)$

$$= \sum_i A_i \delta_{ij}$$

"Kronecker delta" $= 0 \quad i \neq j$

$= 1 \quad i = j$

$$= A_j$$

$$\Longrightarrow \quad A_i = \vec{V}_i \cdot \vec{F}$$

Example: $A_1 = \vec{V}_1 \cdot \vec{F} = \hat{x} \cdot (2\hat{x} + 3\hat{y}) = 2\cancelto{1}{(\hat{x} \cdot \hat{x})} + 3\cancelto{0}{(\hat{x} \cdot \hat{y})} = 2$

We can also find coefficients in a <u>different</u> basis! For example, if $\vec{V}_1 = \frac{\hat{x} + \hat{y}}{\sqrt{2}}$, $\vec{V}_2 = \frac{\hat{x} - \hat{y}}{\sqrt{2}}$, then in this basis,

$A_1 = \vec{V}_1 \cdot \vec{F} = \left(\frac{\hat{x} + \hat{y}}{\sqrt{2}}\right)(2\hat{x} + 3\hat{y}) = \frac{5}{\sqrt{2}}$, $A_2 = \vec{V}_2 \cdot \vec{F} = \frac{\hat{x} - \hat{y}}{\sqrt{2}}(2\hat{x} + 3\hat{y}) = -\frac{1}{\sqrt{2}}$

Determine Coefficients

functions $F(x) = \int_{-\infty}^{\infty} A(k)\, u_k(x)\, dk$

take inner product w/ another basis function:

$$\int_{-\infty}^{+\infty} u_{k'}^*(x)\, F(x)\, dx = \int_{-\infty}^{+\infty} u_{k'}^*(x) \left[\int_{-\infty}^{\infty} A(k)\, u_k(x)\, dk \right] dx$$

$$= \int_{-\infty}^{\infty} A(k) \left[\int_{-\infty}^{\infty} u_{k'}^*(x)\, u_k(x)\, dx \right] dk$$

$$= \int_{-\infty}^{\infty} A(k)\, \delta(k-k')\, dk$$

$$= A(k')$$

"Dirac delta fn":

$\delta(k-k') = 0 \quad k \neq k'$

But $\int_{-\infty}^{\infty} \delta(k-k')\, dk = 1$

$$\Longrightarrow \quad A(k) = \int_{-\infty}^{\infty} u_k^*(x)\, F(x)\, dx$$ "Fourier transform" of $F(x)$

This formula now allows us to solve our P.D.E., given initial conditions!

Diffusion Equation, revisited

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$ solution: use F.T. $$f(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k,t)\, e^{ikx}\, dk$$

$$\int_{-\infty}^{\infty}\frac{e^{-ik'x}}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\dot{\tilde{f}}(k,t)\, e^{ikx}\, dk\right]dx = \int_{-\infty}^{\infty}\frac{e^{-ik'x}}{\sqrt{2\pi}}\left[-\frac{D}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k^2\tilde{f}(k,t)\, e^{ikx}\, dk\right]dx$$

$$\int_{-\infty}^{\infty}\dot{\tilde{f}}(k,t)\left[\int_{-\infty}^{\infty}\frac{e^{-ik'x}}{\sqrt{2\pi}}\frac{e^{ikx}}{\sqrt{2\pi}}\,dx\right]dk = -D\int_{-\infty}^{\infty}k^2\tilde{f}(k,t)\left[\int_{-\infty}^{\infty}\frac{e^{-ik'x}}{\sqrt{2\pi}}\frac{e^{ikx}}{\sqrt{2\pi}}\,dx\right]dk$$

$$\int_{-\infty}^{\infty}\dot{\tilde{f}}(k,t)\,\delta(k-k')\,dk = -D\int k^2\tilde{f}(k,t)\,\delta(k-k')\,dk$$

$$\dot{\tilde{f}}(k',t) = -Dk'^2\tilde{f}(k',t)$$ an O.D.E we know how to solve!

$$\tilde{f}(k,t) = A(k)\, e^{-Dk^2 t}$$

Then,

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)\, e^{-Dk^2 t}\, e^{ikx}\, dk$$ Same as solution by separation of variables!

Closed-form solution of Diffusion Eqn

$$f(x,t=0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk \quad \xleftrightarrow[\text{Transform}]{\text{Fourier}} \quad A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t=0)\, e^{-ikx}\, dx$$

Example: $f(x,t=0) = \delta(x)$ "impulse" initial conditions

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}}$$

So,

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty} A(k) e^{-Dk^2t} e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dk^2t} e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dt\left(k^2 - \frac{ikx}{Dt}\right)}\, dk$$

"Complete the square":

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dt\left(k-\frac{ix}{2Dt}\right)^2} e^{-\frac{x^2}{4Dt}}\, dk$$

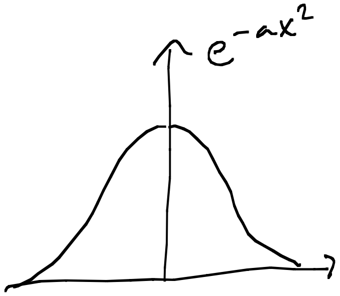
$$= \frac{1}{2\pi} e^{-\frac{x^2}{4Dt}} \int_{-\infty}^{\infty} e^{-Dt\left(k-\frac{ix}{2Dt}\right)^2}\, dk$$

$\sqrt{\pi}$ (see next page)

transform variables $y = \sqrt{Dt}\left(k - \frac{ix}{2Dt}\right)$, $dk = \frac{dy}{\sqrt{Dt}}$:

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4Dt}} \left[\int_{-\infty}^{\infty} e^{-y^2}\, dy\right] \frac{1}{\sqrt{Dt}}$$

Integrating the Gaussian

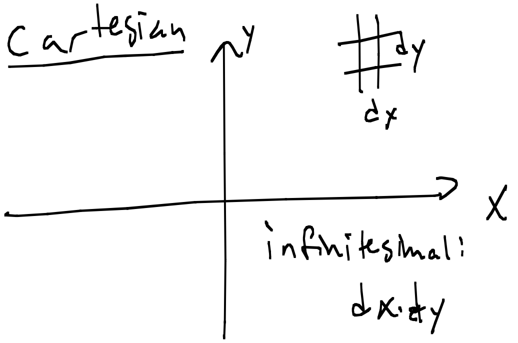


$$I_1 = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$$ How to do this integral?

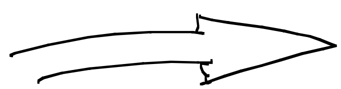
$$I_2 = \int_{-\infty}^{\infty} e^{-ay^2}\,dy$$ Note this is the same, so $I_1 = \sqrt{I_1 \cdot I_2}$

$$I_1 \cdot I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ax^2} e^{-ay^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dx\,dy$$

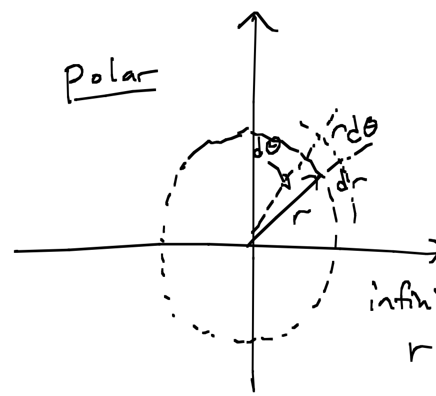
Cartesian



Coordinate transformation



Polar



$x^2 + y^2 = r^2$

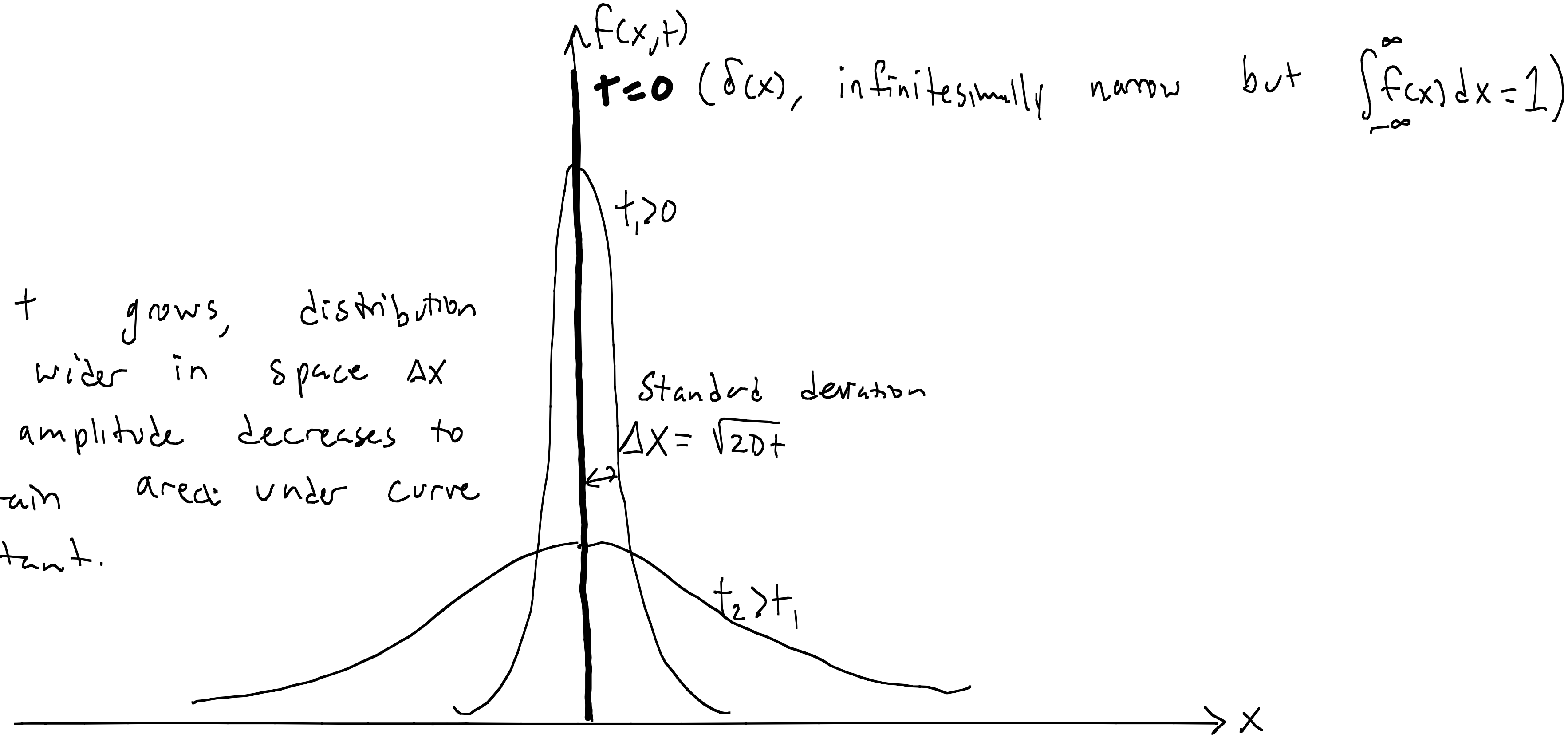
$$I_1 \cdot I_2 = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\,r\,dr\,d\theta = 2\pi \int_0^{\infty} r e^{-ar^2}\,dr = 2\pi \left. \frac{e^{-ar^2}}{-2a} \right|_0^{\infty} = \frac{\pi}{a}$$

So $I_1 = \sqrt{\frac{\pi}{a}}$. Note this can be used to normalize: $\int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}}\, e^{-ax^2}\,dx = 1$

Solution to Diffusion Eqn w/ impulse initial conditions

$$f(x,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{\frac{-x^2}{4Dt}}$$ Normalized "Gaussian" in x

As t grows, distribution gets wider in space Δx and amplitude decreases to maintain area under curve constant.



In general, normalized gaussian has form $f(x) = \frac{1}{\Delta x\sqrt{2\pi}}\, e^{-\frac{x^2}{2\Delta x^2}}$ such that

$$\Delta x = \sqrt{\Delta x^2} = \sqrt{\int_{-\infty}^{\infty} (x - \langle x \rangle)^2 f(x)\, dx}$$ (note that $\langle x \rangle = 0$)