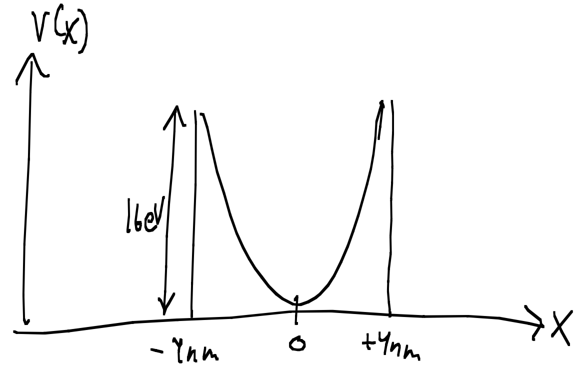
Example: parabolic scatterer

$$V(x) = \begin{cases} 0, & x < -4\,nm \\ \alpha x^2 & -4\,nm < x < 4\,nm \\ 0, & x > 4\,nm \end{cases}$$

$$\alpha = 10^{14}\,eV/cm^2$$



Qualitatively predict $T(E)$ for $E < 16\,eV$ (below top of barriers):

Tunneling into quasi-bound states similar to $V(x) = \frac{1}{2} m \omega^2 x^2$ (SHO).

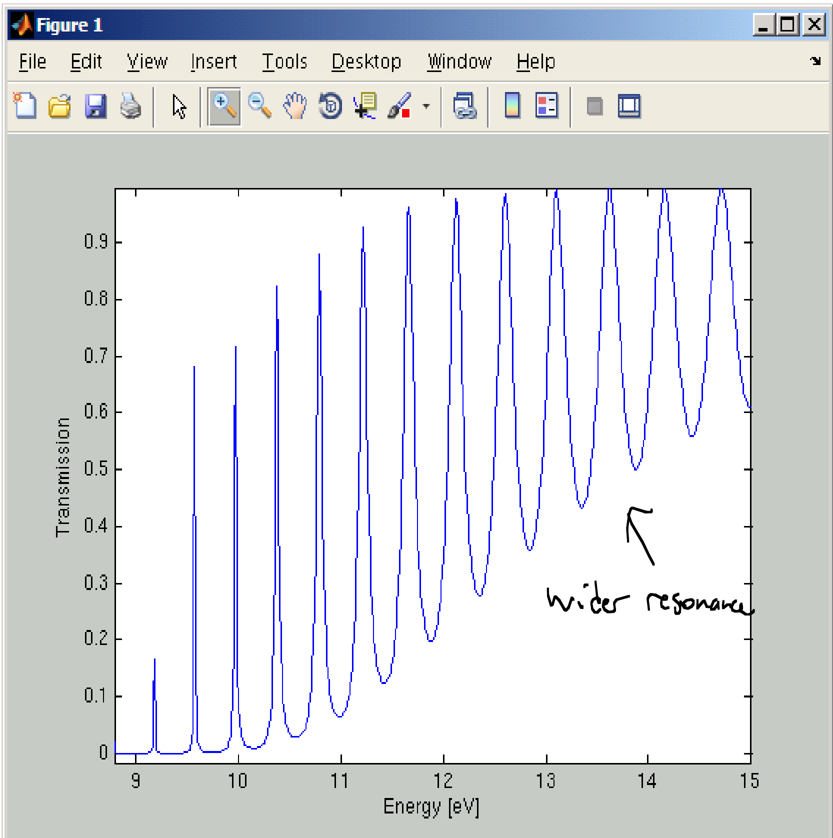
Bound states of SHO are $E = \hbar\omega(n + \frac{1}{2})$ $n = 0, 1, 2 \ldots$

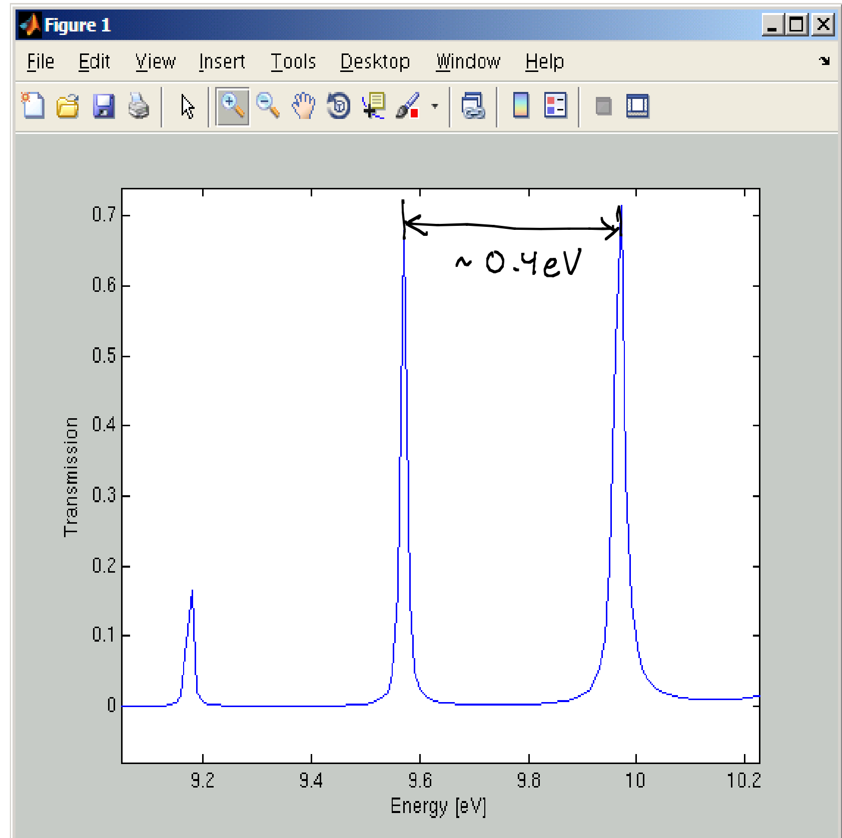
Where $\alpha = \frac{1}{2} m \omega^2$ $\longrightarrow$ $\omega = \sqrt{\frac{2\alpha}{m}} \sim \sqrt{\frac{2 \times 10^{14}\,eV/cm^2}{5.7 \times 10^{-16}\,eV s^2/cm^2}} \sim 5.9 \times 10^{14}\,rad/s \rightarrow \hbar\omega = 0.4\,eV$

So we expect a series of resonances equally spaced by $\sim 0.4\,eV$

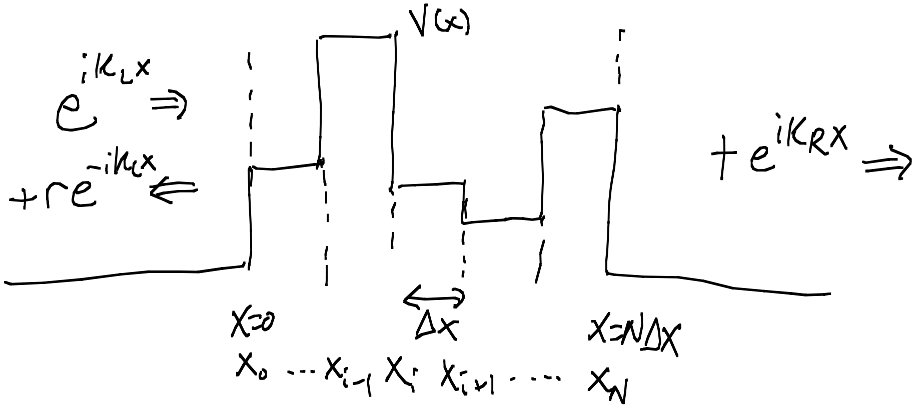
Each resonance will get wider as E increases due to increased coupling to propagating states.

Numerical result





Scattering: Finite differences



Schrödinger Equation

$$\hat{H}\psi = E\psi$$

$$(E\hat{I} - \hat{H})\psi = 0$$

For all interior points $(x \neq 0, N\Delta x)$ $\quad (\psi_i \equiv \psi(x_i))$

$$E\psi_i + \frac{\hbar^2}{2m\Delta x^2}\left(\psi_{i-1} - 2\psi_i + \psi_{i+1}\right) - V_i\psi_i = 0$$

At $x=0$ (left boundary)

$$E\psi_0 + \frac{\hbar^2}{2m\Delta x^2}\left(\text{``}\psi_{-1}\text{''} - 2\psi_0 + \psi_1\right) - V_0\psi_0 = 0$$

$$\text{``}\psi_{-1}\text{''} = e^{ik_L(-\Delta x)} + re^{-ik_L(-\Delta x)} = e^{-ik_L\Delta x} + re^{ik_L\Delta x} = e^{-ik_L\Delta x} + (\psi_0 - 1)e^{ik_L\Delta x}$$

(since $1 + r = \psi_0$ by continuity)

$$E\psi_0 + \frac{\hbar^2}{2m\Delta x^2}\left(e^{-ik_L\Delta x} + (\psi_0 - 1)e^{ik_L\Delta x} - 2\psi_0 + \psi_1\right) - V_0\psi_0 = 0$$

at $x = N\Delta x$ (right boundary)

$$E\varphi_N + \frac{\hbar^2}{2m\Delta x^2}\left(\varphi_{N-1} - 2\varphi_N + \text{``}\varphi_{N+1}\text{''}\right) - V_N\varphi_N = 0$$

$$\text{``}\varphi_{N+1}\text{''} = t\,e^{ik_R(N+1)\Delta x} = e^{ik_R\Delta x}\varphi_N \qquad \left(\text{since } t\,e^{ik_R N\Delta x} = \varphi_N \text{ by continuity}\right)$$

$$E\varphi_N + \frac{\hbar^2}{2m\Delta x^2}\left(\varphi_{N-1} - 2\varphi_N + e^{ik_R\Delta x}\varphi_N\right) - V_N\varphi_N = 0$$

$$\left[E\hat{I} - \begin{bmatrix} 2C + V_0 - Ce^{ik_L\Delta x} & -C & 0 & \cdots & & & \\ & & \ddots & & & & \\ 0 & -C & 2C + V_i & -C & 0 & \cdots & \\ & & & \ddots & & & \\ & \cdots & & 0 & -C & 2C + V_N - Ce^{ik_R\Delta x} \end{bmatrix}\right] \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \vdots \\ \varphi_N \end{bmatrix} = \begin{bmatrix} C\left(e^{ik_L\Delta x} - e^{-ik_L\Delta x}\right) \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \qquad \left(C \equiv \frac{\hbar^2}{2m\Delta x^2}\right)$$

this can be written

$$\left(E\hat{I} - \hat{H}'\right)\vec{\Psi} = \vec{Q}$$ (c.f. bound-state problems where "source" $Q=0$)

Note, however, that the "Hamiltonian" H' is <u>NOT</u> Hermitian, due to the extra terms ($-Ce^{ik_{L/R}\Delta x}$ "self-energy") at the extremes of the main diagonal!

This means that, in general, the eigenvalues of H' <u>are not real!</u>

As a result, the time dependence of Ψ $\left(\phi(t) = e^{-i\frac{E}{\hbar}t}\right)$ is not oscillatory but rather is exponentially decreasing.... There are no truly "<u>bound</u>" or "stationary" states that couple to propagating states! The incoming plane wave interacts w/ the scattering potential, and then after some time $\left(\sim \frac{-\hbar}{\mathrm{Imag}\{E\}}\right)$, leaves either to the left (reflected) or the right (transmitted)

$\longrightarrow$ Particle Probability is <u>not</u> conserved!