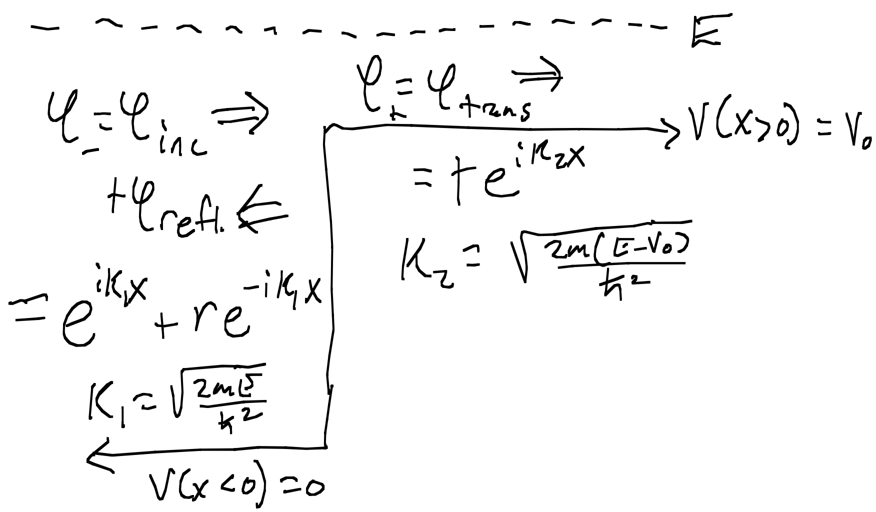
Scattering States

$E > V(x)$ as $x \to \pm\infty$



How much probability flux is transmitted/reflected?

Piecewise-Constant $V(x)$: solutions to time-independent Schrodinger eqn are planewaves.

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi \Rightarrow \psi'' = -k^2\psi, \quad \left(k = \sqrt{\frac{2m(E-V)}{\hbar^2}}\right)$$

Boundary Conditions

1. $\psi_-(0) = \psi_+(0) \to 1 + r = t$

2. $\psi_-'(0) = \psi_+'(0) \to ik_1 - ik_1 r = ik_2 t = ik_2(1+r) \longrightarrow k_1 - k_1 r = k_2 + k_2 r$

$$\to r = \frac{k_1 - k_2}{k_1 + k_2} \qquad \text{Then} \qquad t = 1 + r = \frac{k_1 + k_2}{k_1 + k_2} + \frac{k_1 - k_2}{k_1 + k_2} = \frac{2k_1}{k_1 + k_2}$$

Transmission and Reflection Coefficients

$$J = \frac{\hbar}{m} \, \mathrm{Im} \left\{ \psi^* \frac{d}{dx} \psi \right\}$$

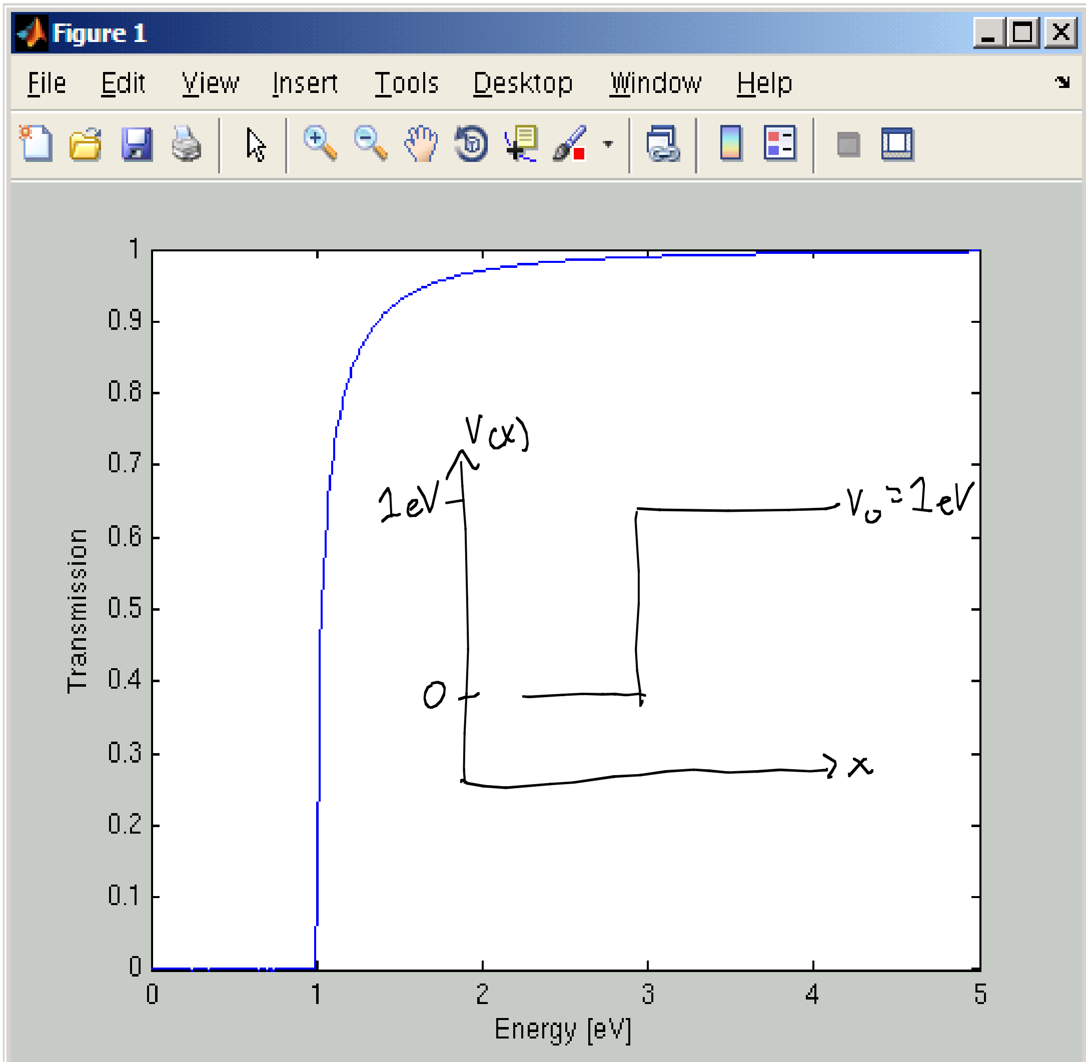
For planewave $\propto e^{ikx}$, $\quad J = \frac{\hbar k}{m} \psi^* \psi$

transmission: $$T = \frac{J_{trans}}{J_{inc}} = \frac{\frac{\hbar k_2}{m} \left(t^* e^{-ik_2 x} \cdot t e^{ik_2 x} \right)}{\frac{\hbar k_1}{m} \left(e^{-ik_1 x} \cdot e^{ik_1 x} \right)} = \frac{k_2}{k_1} |t|^2 = \frac{k_2}{k_1} \frac{4k_1^2}{(k_1+k_2)^2} = \frac{4k_1 k_2}{(k_1+k_2)^2}$$

reflection: $$R = \frac{J_{refl}}{J_{inc}} = \frac{\frac{\hbar k_1}{m} |r|^2}{\frac{\hbar k_1}{m}} = |r|^2 = \frac{(k_1 - k_2)^2}{(k_1+k_2)^2}$$

$$T + R = \frac{4k_1 k_2 + k_1^2 - 2k_1 k_2 + k_2^2}{(k_1+k_2)^2} = \frac{(k_1+k_2)^2}{(k_1+k_2)^2} = 1 \quad \checkmark$$

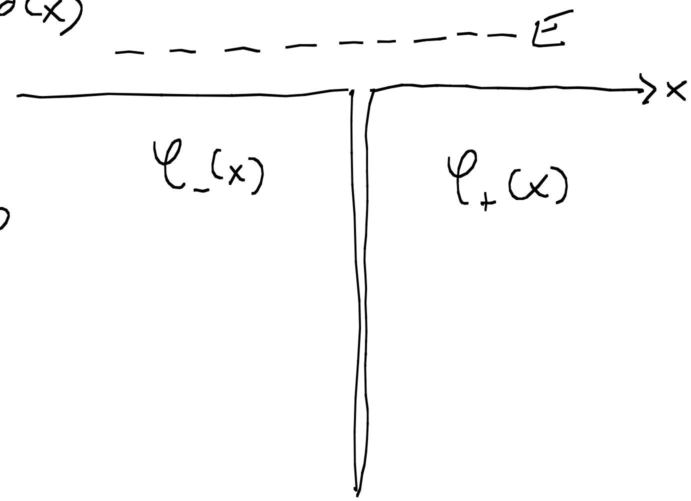
Transmission vs. Energy



Note $T < 1$ above barrier!

(classical mechanics would of course give $T = 1$ for $E > V_0$)

Scattering from divergent potentials $V(x) = -\alpha\delta(x)$



I. Determine $\varphi(x) = \begin{cases} \varphi_-(x) = e^{ikx} + r e^{-ikx}, & x<0 \\ \varphi_+(x) = t e^{ikx}, & x>0 \end{cases}$

$\left(k = \sqrt{\frac{2mE}{\hbar^2}}\right)$

II. Apply B.C.'s to each interface

① $\varphi_-(0) = \varphi_+(0)$

② $\varphi_+'(0) - \varphi_-'(0) = -\frac{2m\alpha}{\hbar^2}\varphi_{+/-}(0)$ (from integration of T.I.S.E.)

III. Calculate $T = J_{trans}/J_{inc}$

Applying Boundary Conditions

(1) $1 + r = t$

(2) $ikt - (ik - ikr) = -\frac{2m\alpha}{\hbar^2}(1+r)$

$$ik(\cancel{1}+r) - ik(\cancel{1}-r) = -\frac{2m\alpha}{\hbar^2}(1+r)$$

$$\cancel{2}ikr = -\frac{\cancel{2}m\alpha}{\hbar^2}(1+r)$$

$$r\left(ik + \frac{m\alpha}{\hbar^2}\right) = -\frac{m\alpha}{\hbar^2}$$

$$r = \frac{-\frac{m\alpha}{\hbar^2}}{ik + \frac{m\alpha}{\hbar^2}} = \frac{\frac{m\alpha}{\hbar^2 k}i}{1 - \frac{m\alpha}{\hbar^2 k}i} = \frac{i\beta}{1 - i\beta} \qquad \left(\beta \equiv \frac{m\alpha}{\hbar^2 k}\right)$$

$$t = 1 + r = \frac{(1 - i\beta) + i\beta}{1 - i\beta} = \frac{1}{1 - i\beta}$$

III. $T = \dfrac{\frac{\hbar k}{m}|t|^2}{\frac{\hbar k}{m}} = |t|^2 = \dfrac{1}{1+i\beta}\,\dfrac{1}{1-i\beta} = \dfrac{1}{1+\beta^2}$

Due to dependence on α^2, Same $T(E)$ for repulsive δ-fn!