Review: Differential Equations

Ordinary differential equations: solution is a function of one variable

"special examples": Can be solved by inspection

$$\frac{df(x)}{dx} = C \longrightarrow f(x) = Cx + A$$

$$\frac{df(x)}{dx} = Cf(x) \longrightarrow f(x) = Ae^{cx}$$

"first order"

one undetermined constant in general sol'n

$$\frac{d^2f(x)}{dx^2} = C \longrightarrow f(x) = \frac{c}{2}x^2 + A_1 x + A_2$$

$$\frac{d^2f(x)}{dx^2} = -c^2 f(x) \longrightarrow f(x) = A_1 \cos(cx) + A_2 \sin(cx)$$

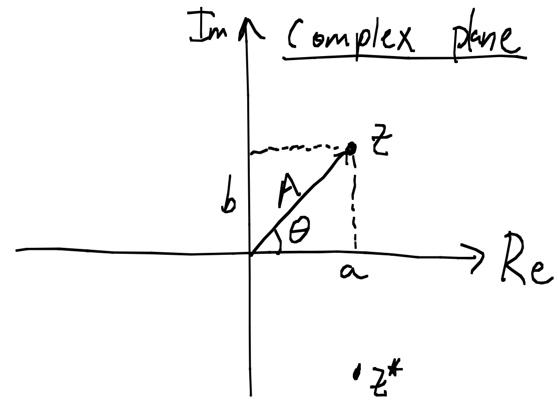
"Second order"

two undetermined constants

OR (mathematically equivalent)

$$= B_1 e^{icx} + B_2 e^{-icx} \qquad \left(i = \sqrt{-1}\right)$$

Complex Numbers



$z = a + ib$ (Cartesian)

$= Ae^{i\theta}$ (polar)

$z^* = a - ib$ "complex conjugate"

$= Ae^{-i\theta}$

Connection between polar and Cartesian:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{"Euler's formula"}$$

proof: Taylor series $f(\theta) = f(0) + f'(0)\,\theta + \frac{f''(0)}{2!}\theta^2 + \frac{f'''(0)}{3!}\theta^3 + \dots$

So $f(\theta) = e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} + i\frac{\theta^3}{3!} + = \left(1 - \frac{\theta^2}{2!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \dots\right) = \cos\theta + i\sin\theta$

Consequences:

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\qquad$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$e^{i\pi} + 1 = 0$ $\qquad$ So "beautiful"!

Solving diff. Eqns. by "Brute Force"

Example:

$$\frac{df}{dx} = Cf \qquad \text{guess: } f(x) = A_0 + A_1 x + A_2 x^2 + \cdots$$

Substitute:

$$A_1 + 2A_2 x + 3A_3 x^2 + \ldots = CA_0 + CA_1 x + CA_2 x^2 + \cdots$$

Since this must be true for all x, equate like powers of x:

$$(x^0): \quad A_1 = CA_0,$$

$$(x^1): \quad 2A_2 = CA_1 = C^2 A_0 \quad \rightarrow \quad A_2 = \frac{A_0 C^2}{2}$$

$$(x^2): \quad 3A_3 = CA_2 = \frac{C^3 A_0}{2} \quad \rightarrow \quad A_3 = \frac{A_0 C^3}{3 \cdot 2}$$

$$\vdots$$

So:

$$f(x) = A_0 + A_0 Cx + A_0 \frac{C^2 x^2}{2} + A_0 \frac{C^3 x^3}{3 \cdot 2} + \cdots$$

$$= A_0 \left[1 + (Cx) + \frac{(Cx)^2}{2!} + \frac{(Cx)^3}{3!} + \ldots \right] = A_0 e^{Cx}$$

Compare w/ sol'n by inspection!

A simpler route to solution by "brute force"

Solve $f' = \frac{df}{dx} = cf$ by substituting infinite sum $f = \sum_{n=0}^{\infty} A_n x^n$

$$\sum_{n=0}^{\infty} n A_n x^{n-1} = c \sum_{n=0}^{\infty} A_n x^n$$

$$\sum_{n=1}^{\infty} n A_n x^{n-1} = \sum_{n=0}^{\infty} c A_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) A_{n+1} x^n = \sum_{n=0}^{\infty} c A_n x^n$$

$$A_{n+1} = \frac{c}{n+1} A_n \qquad \text{"recursion relation"}$$

$$f = A_0 \sum_{n=0}^{\infty} \frac{c^n}{n!} x^n = A_0 \sum_{n=0}^{\infty} \frac{(cx)^n}{n!} = A_0 e^{cx} \qquad \checkmark$$